Supersymmetric quantum mechanics and its applications

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Abstract. The Hamiltonian in Supersymmetric Quantum Mechanics is defined in terms of charges that obey the same algebra as that of the generators of supersymmetry in field theory. The consequences of this symmetry for the spectra of the component parts that constitute the supersymmetric system are explored. The implications of supersymmetry for the solutions of the Schrödinger equation, the Dirac equation, the inverse scattering theory and the multi-soliton solutions of the KdV equation are examined. Applications to scattering problems in Nuclear Physics with specific reference to singular potentials which arise from considerations of supersymmetry will be discussed.

1. SUPERSYMMETRIC QUANTUM MECHANICS OF ONE-DIMENSIONAL SYSTEMS

It is shown that every one-dimensional quantum mechanical Hamiltonian $H$ can have a partner $\tilde{H}$ such that $H$ and $\tilde{H}$ taken together may be viewed as the components of a supersymmetric Hamiltonian $H$. The term ‘supersymmetric Hamiltonian’ is taken to mean a Hamiltonian defined in terms of charges that obey the same algebra as that of the generators of supersymmetry in field theory. The consequences of this symmetry for the spectra of $H$ and $\tilde{H}$ are explored. It is shown how the supersymmetric pairing may be used to eliminate the ground state of $H$, or add a state below the ground state of $H$ or maintain the spectrum of $H$. It is also explicitly demonstrated that the supersymmetric pairing may be used to generate a class of anharmonic potentials with exactly specified spectra.

1.1. Introduction

In field theory, supersymmetry is a symmetry that generates transformations between bosons and fermions. Unlike the generators of other symmetries whose algebra involves commutators, the generators of the supersymmetric transformations are spinor charges whose algebra involves anticommutators. Supersymmetry has raised the possibility of providing a framework for a unified description of bosons and fermions which are combined in the same supersymmetric multiplet [1]. Supersymmetric field theories may be constructed by defining a superfield in a superspace, a space consisting of the usual spacetime and in addition the anticommuting spinors of Grassmann [2]. The superfield $\phi$ is a function of the spacetime coordinates $x$ and also $\theta$ and $\bar{\theta}$ where $\theta$ is an odd member of the Grassmann algebra and $\bar{\theta}$ is its conjugate. The supersymmetric transformation
may be viewed as a Grassmann even translation in this superspace. The generators of this transformation are the supercharges.

There is a well-defined procedure for starting from a field theory to construct a single particle Quantum Mechanics and vice versa. For example Quantum Mechanics in one dimension is defined by the Hamiltonian \( H = p^2 + V(x) \) and the commutation relation \([x, p] = i\hbar\). The corresponding field theory starts by defining a space-time and the field \( \phi(x, t) \) is defined in this space-time by the Lagrangian \( L = ((\partial_t \phi)^2 - V(\phi)) \) and the action \( S = \int L dt \). It is well known that \( d = 1 \) quantum mechanics is formally equivalent to the \( d = 1 \) quantum field theory with the identification \( x \rightarrow \phi, p \rightarrow \partial_t \phi \) and canonical quantization of the field \( \phi \) leads to the usual commutation relations between \( x \) and \( p \). Similarly, by constructing a Lagrangian invariant under the supersymmetric transformation, \( i.e. \) by generalizing the \( d = 1 \) field to the superfield defined in superspace and integrating out the Grassmann coordinates associated with the superspace, a Lagrangian expressed in terms of the component fields of the superfield may be obtained. Canonical quantization then leads to a Hamiltonian for Supersymmetric Quantum Mechanics. Witten [3] was the first to construct a simple example of a supersymmetric system corresponding to a spin- \( \frac{1}{2} \) particle moving in one dimension. Witten also defined the algebra that must be satisfied by the charge operators in terms of which the supersymmetric Hamiltonian may be expressed. These algebraic relations that Witten first formulated have now become the defining relations of Supersymmetric Quantum Mechanics or SUSYQM in abbreviation.

The word ‘supersymmetry’ was originally used to denote a symmetry built into certain field theories that permits transformations between component fields whose intrinsic spins differ by \( \frac{1}{2} \hbar \). However, by extracting a single particle Quantum Mechanics from the field theory by integration of the Grassmann variables all reference to spin is lost. What remains is an underlying symmetry of the Schrödinger differential equations for two related Hamiltonians. In fact, already in the 19th century a symmetry of second-order differential equations had been identified by Darboux [4]. The Darboux transformation relates the solutions of a pair of closely linked second order differential equations (Schrödinger [5], Infeld and Hull [6]). Throughout these lectures the term ‘supersymmetric system’ will be used to describe systems governed by an underlying algebra which is identical to, or derivable from, the algebra of supersymmetry in field theory even if the systems under consideration have nothing to do with bosons and fermions as they are commonly understood. This algebra is the algebra explicitly defined by Witten.

The study of the relationship between spectra, conservation laws and the existence of operators that commute with the Hamiltonian has a long history. It is well known that the conservation of energy, linear momentum and angular momentum arise when space-time is homogeneous and isotropic which in turn lead to invariance of the Hamiltonian under time translation, spatial translation and rotation. The invariance of the supersymmetric Hamiltonian under translations in superspace is related to the existence of supercharges that commute with the SUSY Hamiltonian and leads to definite relations between the spectra of the bosonic and fermionic sectors. Just as in field theory supersymmetry leads to specific relations between the component sectors of a supermultiplet, so also in SUSYQM the existence of a generating operator that commutes with the Hamiltonian leads to certain specific relations between the spectra and the eigenfunctions of the component parts of the supersymmetric Hamiltonian. The links between the solutions
of two differential equations connected by the Darboux transformation are identical to those arising from considerations of supersymmetry. Witten’s seminal idea has now been developed into the subject of Supersymmetric Quantum Mechanics: the study of quantum mechanical systems governed by an algebra identical to that of supersymmetry in field theory. A number of people have played an important role in the development of the subject. It will not be possible in these lectures to do full justice to all the people who have contributed to this subject. I would like to keep a chronological order of how the ideas have developed and refer to the papers that act as markers in this progression. Witten’s 1981 paper was followed by other examples of spin systems in magnetic fields and other such special systems that exhibited an underlying supersymmetry ([7]-[14]). Bernstein and Brown [15] showed that by exploiting the degeneracy between the ‘bosonic’ and ‘fermionic’ sectors of certain one-dimensional Hamiltonians, the properties of the first excited state of the ‘bosonic’ component may be inferred from a knowledge of the ground state of the ‘fermionic’ component. It was then shown by Andrianov, Borisov and Ioffe [16] and Sukumar [17, 18] that all one-dimensional systems can have supersymmetric partners. Andrianov, Borisov and Ioffe [19] also showed that a simple extension of supersymmetric quantum mechanics to arbitrary dimensions is possible.

The plan of the lectures is to cover topics in the following order. In the first lecture the defining algebra of Supersymmetric Quantum Mechanics, the implications of this algebra for the spectra of the component parts of the SUSY Hamiltonian, the factorization of the Schrödinger equation, the procedure for the elimination of the ground state of a Hamiltonian, the procedure for the introduction of a new bound state below the ground state of a given Hamiltonian and the procedure for generating a new Hamiltonian with unaltered spectrum will be discussed using examples. Thus the first lecture will mainly be concerned with showing that the existence of a SUSY partner to one-dimensional Hamiltonians implies a hierarchy of Hamiltonians with a special relationship between the eigenvalues and eigenfunctions of the different members of the hierarchy.

In the second lecture the radial Schrödinger equation will be studied. The implications of a SUSY partner to the radial Schrödinger equation will be used to differentiate between four types of SUSY transformations. In the first part of the third lecture the underlying supersymmetry of the Dirac equation for the Hydrogen atom will be discussed. In the second part of the third lecture the supersymmetry linking the $N$ and the $N+1$ soliton solutions of the KdV equation will be discussed. The connection between the different types of supersymmetric transformations of the radial Schrödinger equation and the approach of the conventional inverse scattering theory will be more fully explored in the fourth and fifth lectures. It will be shown how certain choices of pairs of SUSY transformations lead to the results of the conventional inverse scattering theory based on the Gelfand-Levitan and Marchenko equations. It will be shown that other choices of pairs of SUSY transformations lead to new results not present in the standard inverse scattering theory and produce singular potentials which have found a variety of applications in Nuclear Physics. The fourth lecture will be concerned with the study of different pairs of supersymmetric transformations and the different potentials that can be generated by this procedure. In the fifth lecture the procedures discussed in the earlier lectures will be generalized and some applications of the new aspects of inverse scattering theory arising from the singular potentials constructed using SUSY transformations of the
radial Schrödinger equation will be discussed. Throughout these lectures units in which $\hbar = 1$ and the mass $m = 1$ will be used.

1.2. Supersymmetric quantum mechanics

SUSYQM is characterized by the existence of charge operators $Q_i$ that obey the algebra

$$\{Q_i, Q_j\} = \delta_{ij} \mathbf{H}, \quad i, j = 1, 2, \ldots, N, \quad \{Q_i, \mathbf{H}\} = 0,$$

(1)

(2)

where $\mathbf{H}$ is the supersymmetric Hamiltonian, $N$ is the number of generators and $\{,\}$ denotes an anticommutator. Here we consider the simplest of such systems with two operators $Q_1$ and $Q_2$. In terms of $Q = (Q_1 + iQ_2)/\sqrt{2}$ and its Hermitian adjoint $Q^\dagger = (Q_1 - iQ_2)/\sqrt{2}$ the algebra governing this supersymmetric system is characterized by

$$\mathbf{H} = \{Q, Q^\dagger\}, \quad Q^2 = 0, \quad Q^{12} = 0.$$

(3)

From these equations it is clear that

$$[Q, \mathbf{H}] = 0, \quad [Q^\dagger, \mathbf{H}] = 0.$$

(4)

i.e., the charge operator $Q$ is nilpotent and commutes with the Hamiltonian $\mathbf{H}$. A simple realization of the algebra defined in Eq. (3) can be achieved by considering

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix},$$

(5)

where $A^-$ is an operator and $A^+$ is its adjoint. It is clear that with this construction $Q^2 = 0$ automatically. Eqs. (3) and (5) lead to the supersymmetric Hamiltonian

$$\mathbf{H} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix}.$$

(6)

Since

$$Q \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A^-\alpha \end{pmatrix}, \quad Q^\dagger \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} A^+\beta \\ 0 \end{pmatrix},$$

(7)

we can say that the operators $Q$ and $Q^\dagger$ induce transformations between the ‘bosonic’ sector represented by $\alpha$ and the ‘fermionic’ sector represented by $\beta$. We may also interpret $\mathbf{H}$ in the following way: the scalar Hamiltonian $H = A^+A^-$ has a partner $\tilde{H} = A^-A^+$ such that $H$ and $\tilde{H}$ are the diagonal elements of a supersymmetric Hamiltonian $\mathbf{H}$. Having demonstrated that a $Q$ and an $\mathbf{H}$ can be constructed, we can switch to the operator language of Quantum Mechanics to find out what the consequences of the existence of a charge operator that commutes with the Hamiltonian are for the spectra of the two sectors $H$ and $\tilde{H}$. $A^+A^-$ and $A^-A^+$ are both positive semi-definite operators.
with eigenvalues greater than or equal to 0. Let $\psi$ be a normalized eigenstate of $H$ with eigenvalue $E$. Then

$$A^+A^- \psi = E \psi.$$  \hfill (8)

Multiplication from the left by $A^-$ leads to

$$A^-A^+(A^-\psi) = E(A^-\psi).$$  \hfill (9)

If $A^-\psi \neq 0$, we can infer that $E$ is also an eigenvalue of $A^-A^+$. The corresponding normalized eigenfunction $\tilde{\psi}$ of $\tilde{H}$ can be shown to be given by

$$\tilde{\psi} = \frac{1}{\sqrt{E}} (A^-\psi).$$  \hfill (10)

The same reasoning may be applied starting from the eigenvalue equation for $\tilde{H}$ instead of $H$ to investigate whether every eigenvalue of $\tilde{H}$ is also an eigenvalue of $H$. If $\tilde{E}$ is an eigenvalue of $\tilde{H}$ with eigenfunction $\tilde{\psi}$

$$A^-A^+ \tilde{\psi} = \tilde{E} \tilde{\psi},$$

then

$$A^+A^- (A^+\tilde{\psi}) = \tilde{E}(A^+\tilde{\psi}).$$  \hfill (12)

Therefore, if $A^+\tilde{\psi} \neq 0$ then $\tilde{E}$ is also an eigenvalue of $H$ with the corresponding normalized eigenfunction

$$\psi = \frac{1}{\sqrt{\tilde{E}}}(A^+\tilde{\psi}).$$  \hfill (13)

In view of the above relationships, three possibilities may be distinguished from each other.

(a) If there is a normalizable eigenstate of $H$ such that $A^-\psi^{(0)} = 0$, then $A^+A^-\psi^{(0)} = 0$ and $\psi^{(0)}$ corresponds to the ground state with eigenvalue $E^{(0)} = 0$. Conversely, for the eigenvalue $E^{(0)} = 0$ the vanishing expectation value of $A^+A^-$ in the ground state implies that $A^-\psi^{(0)} = 0$. Under these circumstances, $\tilde{H}$ has no normalizable eigenstate with $\tilde{E} = 0$, i.e. there can be no normalizable state with $A^+\tilde{\psi} = 0$. The ground state eigenvalue of $\tilde{H}$ is non-zero. All eigenvalues other than the ground state eigenvalue of $\tilde{H}$ are also eigenvalues of $\tilde{H}$ and all eigenvalues of $\tilde{H}$ are also eigenvalues of $H$. The resulting spectral mapping is shown in Fig. 1(a).

(b) If there is a normalizable eigenstate of $\tilde{H}$ such that $A^+\tilde{\psi}^{(0)} = 0$, then $A^-A^+\tilde{\psi}^{(0)} = 0$ and $\tilde{\psi}^{(0)}$ corresponds to the ground state of $A^-A^+$ with eigenvalue $\tilde{E}^{(0)} = 0$. There cannot be a normalizable state of $A^+A^-$ with eigenvalue zero that satisfies $A^-\psi = 0$. The ground state of $H$ has non-zero eigenvalue. All eigenvalues other than the ground state of $\tilde{H}$ are also eigenvalues of $H$ and all eigenvalues of $H$ are eigenvalues of $\tilde{H}$. This leads to the spectral mapping shown in Fig. 1(b).

(c) If there is no normalizable eigenstate of $H$ or $\tilde{H}$ such that either $A^-\psi = 0$ or $A^+\tilde{\psi} = 0$, then the spectra of both $H$ and $\tilde{H}$ begin at positive values. Every eigenvalue of $\tilde{H}$ is also an eigenvalue of $H$ and vice versa. The resulting spectral mapping is shown in Fig. 1(c).
In each of the three cases the eigenfunctions of $H$ and $\tilde{H}$ for a common eigenvalue $E$ are linked in the manner indicated below:

$$
\psi(E) = \exp(i\phi)(E)^{-\frac{1}{2}}A^+\psi(E),
$$
$$
\tilde{\psi}(E) = \exp(-i\phi)(E)^{-\frac{1}{2}}A^-\tilde{\psi}(E),
$$

in which $\phi$ is an arbitrary phase whose significance will become clear in later discussion. The ladder structure of the eigenvalue spectrum shown in Fig. 1 and the intertwining relationships between the eigenfunctions given above are characteristic hallmarks of supersymmetric systems in one dimension and serve as signatures by which the presence of an underlying supersymmetry may be inferred. In the early works on SUSYQM ([12, 15]) operators of the form

$$
A^\pm = \left( \pm \frac{d}{dx} + U(x) \right),
$$

were considered in which $U(x)$ was considered to be a known function of $x$. This assumption restricts the applicability of SUSYQM to a limited class of problems. In the next section it will be shown that it is not necessary to make any assumptions about $U$ and that $U$ itself may be generated from the solutions of the Schrödinger equation in one dimension. Such a generalization extends the applicability of SUSYQM to all one-dimensional problems which have a ground state.

### 1.3. Factorization of the Schrödinger equation

The Schrödinger equation in one dimension is governed by the Hamiltonian

$$
H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x),
$$
where $V$ is the potential. $H$ can be factorized in the form

$$H = A^+ A^- + \epsilon , \quad A^\pm = \frac{1}{\sqrt{2}} \left( \pm \frac{d}{dx} + U \right), \quad (17)$$

where $\epsilon$ is an undetermined constant, provided that the unknown function $U$ satisfies

$$\left( \frac{dU}{dx} + U^2 \right) = 2(V - \epsilon). \quad (18)$$

This is a nonlinear equation with a family of solutions. One member of the family is given by

$$U(x, \epsilon) = \frac{1}{\psi(x, \epsilon)} \frac{d}{dx} \psi(x, \epsilon), \quad (19)$$

where $\psi(x, \epsilon)$ is a solution of the Schrödinger equation at energy $E = \epsilon$, i.e.

$$H \psi(x, \epsilon) = \epsilon \psi(x, \epsilon). \quad (20)$$

Since

$$\frac{dU}{dx} = \frac{1}{\psi} \frac{d^2}{dx^2} \psi - \frac{1}{\psi^2} \left( \frac{d\psi}{dx} \right)^2, \quad (21)$$

it is easy to verify that Eq. (19) satisfies Eq. (18). It is clear that this argument is valid only if $\psi(x, \epsilon)$ is non-vanishing i.e. $\psi(x, \epsilon)$ is nodeless. It can be shown that the general solution to Eq. (18) is given by the one-parameter family of solutions

$$U(x, \epsilon, \lambda) = \frac{d}{dx} \ln \psi(x, \epsilon) + \frac{1}{\lambda + \int^x dz/\psi^2(z, \epsilon)}, \quad (22)$$

where $\lambda$ is an arbitrary parameter. Every choice of $\epsilon$ and the corresponding $\psi(x, \epsilon)$ leads to a possible factorization of $H$ in the form $H = A^+(\epsilon)A^-(\epsilon) + \epsilon$. The choice of factorization energy $\epsilon$ and the selection of a value for $\lambda$ must clearly be motivated by the particular circumstances of a given problem and by physical considerations. If we consider a Hamiltonian with a ground state energy $E^{(0)}$, then the requirement that $A^+A^-$ be a positive definite operator can be met only if the energy $\epsilon$ is chosen to be $\epsilon \leq E^{(0)}$. We consider the case when the factorization energy $\epsilon = E^{(0)}$ next.

### 1.4. Factorization energy $\epsilon$ equals the ground state energy $E^{(0)}$

With the choice of $E^{(0)}$ as the factorization energy, the ground state eigenfunction $\psi(x, E^{(0)})$ is nodeless and vanishes in the asymptotic region. The requirement that $U(x)$ in Eq. (22) should not be divergent leads to the choice $\lambda = \infty$, giving

$$U(x) = \frac{d}{dx} \ln \psi(x, E^{(0)}),$$

$$A^\pm(E^{(0)}) = \frac{1}{\sqrt{2}} \left[ \pm \frac{d}{dx} + \frac{d}{dx} \ln \psi(x, E^{(0)}) \right],$$

$$H = A^+(E^{(0)})A^-(E^{(0)}) + E^{(0)}. \quad (23)$$
It is clear that $A^+ A^-$ has a spectrum beginning at 0, with a ground state which satisfies $A^- \psi^{(0)} = 0$ with $\psi^{(0)} = \psi(x, E^{(0)})$. The analysis in §1.3 can now be used by considering the partner Hamiltonian

$$\tilde{H} = E^{(0)} + A^- (E^{(0)}) A^+ (E^{(0)}) = H + [A^- (E^{(0)}), A^+ (E^{(0))}] ,$$

(24)

corresponding to the potential

$$\tilde{V}(x) = V(x) - \frac{d^2}{dx^2} \ln \psi^{(0)} .$$

(25)

$H$ and $\tilde{H}$ have their spectra aligned as in Fig. 1(a). $\tilde{H}$ has no eigenstate corresponding to the ground state of $H$ and all the excited states of $H$ are degenerate with the eigenstates of $\tilde{H}$. The eigenfunctions of the two Hamiltonians are linked in the form

$$\psi(x, E) = (E - E^{(0)})^{-\frac{1}{2}} A^- (E^{(0)}) \psi(x, E) ,$$

$$\psi(x, E) = (E - E^{(0)})^{-\frac{1}{2}} A^+ (E^{(0)}) \tilde{\psi}(x, E) ,$$

(26)

by choosing the phase $\phi$ in Eq. (14) to be zero. These equations are valid not only when $E$ is one of the discrete eigenvalues of $H$ , $E = E^{(j)} (j \neq 0)$, but also when $E$ lies in the continuous part of the spectrum. When $E$ lies in the continuous part of the spectrum of $H$ the above equations can be used to find a relation between the transmission coefficients in the potentials $V(x)$ and $\tilde{V}(x)$ at energy $E$ since the asymptotic form of the wavefunction for potential $V$ at energy $E$ implies a definite asymptotic form for the wavefunction for $\tilde{V}$ at the same energy. This procedure will be illustrated by considering the phase shift for the solutions of the radial Schrödinger equation in the second lecture. Since the above analysis is valid for any one-dimensional Hamiltonian $H_1$ with the ground state with energy $E^{(0)}_1$ and wave function $\psi^{(0)}_1$ the process of finding a supersymmetric partner can be iterated to generate the hierarchy of Hamiltonians given by

$$H_n(x) = -\frac{1}{2} \frac{d^2}{dx^2} + V_n(x) = A_n^+ A_n^- + E_n^{(0)} = A_{n-1} A_{n-1}^+ + E_{n-1}^{(0)} , \quad n = 2, 3, \ldots ,$$

(27)

where

$$A_n^\pm (x) = \frac{1}{\sqrt{2}} \left[ \pm \frac{d}{dx} + \frac{d}{dx} \ln \psi_n^{(0)} (x) \right] , \quad n = 1, 2, \ldots ,$$

$$V_n(x) = V_{n-1}(x) - \frac{d^2}{dx^2} \ln \psi_n^{(0)} (x) , \quad n = 2, 3, \ldots ,$$

(28)

in which $E_n^{(j)}$ and $\psi_n^{(j)}$ are the eigenenergies and eigenfunctions of $H_n$ with the property that

$$E_n^{(m)} = E_{n-1}^{(m+1)} = \ldots = E_{n+m-1}^{(m+1)} ,$$

$$\psi_n^{(m)} = [E_n^{(m)} - E_{n-1}^{(0)}]^{-\frac{1}{2}} A_{n-1}^- \psi_{n-1}^{(m+1)} ,$$

$$\psi_{n-1}^{(m+1)} = [E_n^{(m)} - E_{n-1}^{(0)}]^{-\frac{1}{2}} A_{n-1}^+ \psi_n^{(m)} , \quad n = 2, 3, \ldots , \quad m = 0, 1, 2, \ldots .$$

(29)
A pictorial representation of the eigenvalue correspondence of the Hamiltonian hierarchy is given in Fig. 2.

The equations given above show that the excited states of $V_1$ can be obtained from the ground states of the hierarchy $V_n$. The simple harmonic oscillator, the particle in a box, the radial equation for a definite partial wave for the Coulomb potential and the Morse potential are all examples where the potentials corresponding to the Hamiltonians in the hierarchy can be analytically worked out [17]. One nontrivial example of such an exactly solvable hierarchy will be discussed next.

**1.5. Attractive sech\(^2\) potential**

Let

$$V_1 = -\lambda_1 \text{sech}^2 x, \quad \lambda_1 > 0 .$$  \hspace{1cm} (30)

Since this potential is attractive in all space $-\infty \leq x \leq \infty$ it will support at least one bound state irrespective of the strength of the potential. In terms of the parameter

$$Q_1 = \left( 2\lambda_1 + \frac{1}{4} \right)^{\frac{1}{2}} \geq \frac{1}{2} ,$$  \hspace{1cm} (31)

the spectrum of this potential is given by [20]

$$E_1^{(m)} = -\frac{1}{2} \left[ Q_1 - \left( m + \frac{1}{2} \right) \right] ^2, \quad m = 0, 1, 2, \ldots, N \leq Q_1 - \frac{1}{2} .$$  \hspace{1cm} (32)

The potential in Eq. (30) supports a finite number $(N + 1)$ of bound states. The ground state wavefunction

$$\psi_1^{(0)}(x) \sim \text{sech}^{(Q_1 - \frac{1}{2})} x ,$$  \hspace{1cm} (33)

leads to

$$V_2(x) = -\left( \lambda_1 + \frac{1}{2} - Q_1 \right) \text{sech}^2 x .$$  \hspace{1cm} (34)
Inspection of this equation shows that
(i) if \( \lambda_1 > 1 \), then \( V_2(x) \) is an attractive sech\(^2\)x potential;
(ii) if \( \lambda_1 = 1 \), then \( V_2(x) \) vanishes and \( H_2 \) is a free particle Hamiltonian;
(iii) if \( \lambda_1 < 1 \), then \( V_2(x) \) is a repulsive potential and corresponds to a sech\(^2\)x barrier.

It is easy to show that the parameter corresponding to \( Q_1 \) for \( V_2 \) is
\[
Q_2 = \left[ \frac{1}{4} + 2 \left( \lambda_1 + \frac{1}{2} - Q_1 \right) \right]^{\frac{1}{2}} = Q_1 - 1 .
\] (35)

The spectrum of \( H_2 \) is then given by
\[
E_2^{(m)} = -\frac{1}{2} \left[ Q_2 - \left( m + \frac{1}{2} \right) \right]^2 ,
\] (36)
which satisfies the condition \( E_2^{(m)} = E_1^{(m+1)} \). Iteration of this argument shows that the Hamiltonian hierarchy corresponds to a sequence of shape-invariant potentials with successively decreasing strengths. It is easy to show that
\[
V_n(x) = -\lambda_n \text{sech}^2 x , \quad Q_n = \left( 2\lambda_n + \frac{1}{4} \right)^{\frac{1}{2}} = Q_{n-1} - 1 .
\] (37)

If \( N = Q_1 - \frac{1}{2} \) then \( V_{N+1}(x) \) vanishes. If \( N < Q_1 - \frac{1}{2} < N + 1 \) then \( V_{N+2} \) corresponds to a sech\(^2\) barrier given by
\[
V_{N+2}(x) = \frac{1}{2} \left( Q_1 - N - \frac{1}{2} \right) \left( N + \frac{3}{2} - Q_1 \right) \text{sech}^2 x .
\] (38)

We have shown that by choosing the factorization energy \( \varepsilon \) to be the ground state energy it is possible to generate a new Hamiltonian \( H_2 \) without an eigenstate corresponding to the ground state of \( H_1 \) but retaining the rest of the spectrum of \( H_1 \). It has been demonstrated that this procedure may be iterated to generate a Hamiltonian hierarchy with spectra aligned as in Fig. 2. In the next subsection we examine other possible factorizations.

### 1.6. Factorization energy \( \varepsilon \) less than the ground state energy

When the factorization energy \( \varepsilon \) in Eq. (17) is less than the ground state energy of \( H \) the solution \( \psi(x, \varepsilon) \) of \( H \psi = \varepsilon \psi \) is not a normalizable solution even though \( \psi(\varepsilon) \) is still a solution of \( A^- \psi(\varepsilon) = 0 \). The lack of normalizability of \( \psi(\varepsilon) \) means that \( A^+ A^- \) cannot have zero as an eigenvalue and the spectrum of \( A^+ A^- \) begins at positive values. The analysis in §1.3 shows that when \( A^+ A^- \) has no normalizable eigenstate with eigenvalue zero, it is possible for \( A^- A^+ \) to have a spectrum beginning at eigenvalue zero. For \( A^- A^+ \) to have a normalizable state with eigenvalue zero, the solution \( \tilde{\psi} \) of \( A^+ (\varepsilon) \tilde{\psi} = 0 \) must be normalizable. The solution of
\[
\left[ \frac{d}{dx} + \frac{d}{dx} \ln \psi(x, \varepsilon) \right] \tilde{\psi}(x, \varepsilon) = 0 ,
\] (39)
\[ i.e. \quad \bar{\psi}(x, \varepsilon) = \frac{1}{\psi(x, \varepsilon)}, \quad (40) \]

shows that if the unnormalizable solution \( \psi(x, \varepsilon) \) of the Hamiltonian \( H \) is chosen in such a way that \( (\psi)^{-1} \) is normalizable, then \( \bar{\psi}(x, \varepsilon) \) is normalizable and \( A^{-}A^{+} \) has a spectrum beginning at eigenvalue zero. Therefore

\[ \bar{H} = A^{-}(\varepsilon)A^{+}(\varepsilon) + \varepsilon, \quad \varepsilon < E^{(0)}, \quad (41) \]

has a ground state at energy \( \bar{E}^{(0)} \) with a ground state eigenfunction \( \bar{\psi}^{(0)}(x, \varepsilon) = \bar{\psi}(x, \varepsilon) \). Therefore, \( \bar{H} \) has a ground state eigenvalue below the ground state of \( H \) while all the other eigenvalues of \( \bar{H} \) are degenerate with the eigenvalues of \( H \). This corresponds to the level scheme shown in Fig. 1(b). Hence when \( (\psi)^{-1} \) is normalizable

\[ \bar{H} = H - \frac{d^{2}}{dx^{2}} \ln \psi(x, \varepsilon), \quad (42) \]

has ground state

\[ \bar{E}^{(0)} = \varepsilon < E^{(0)}, \quad \bar{\psi}^{(0)}(x, \varepsilon) = \frac{1}{\psi(x, \varepsilon)}, \quad (43) \]

and excited states with

\[
\begin{align*}
E^{(m+1)} &= E^{(m)}, \quad m = 0, 1, 2, \ldots, \\
\bar{\psi}^{(m+1)} &= -\left( E^{(m)} - \varepsilon \right)^{-\frac{1}{2}} A^{-}\psi^{(m)}, \\
\psi^{(m)} &= -\left( E^{(m)} - \varepsilon \right)^{-\frac{1}{2}} A^{+}\bar{\psi}^{(m+1)}, \\
A^{\pm}(\varepsilon) &= \frac{1}{\sqrt{2}} \left[ \pm \frac{d}{dx} + \frac{d}{dx} \ln \psi(x, \varepsilon) \right], \quad (44)
\end{align*}
\]

The phase factor \( \phi \) in Eq. (14) has been chosen to be \( \pi \). Having chosen \( \phi \) to be zero for the case of elimination of a state in §1.5, the requirement that adding a state by a transformation and subsequently eliminating the same state by another transformation should give back the original transformation, fixes the phase factor for the case of the addition of a state to be \( \pi \). If \( \varepsilon < E^{(0)} \), but the unnormalizable solution \( \psi(x, \varepsilon) \) does not lead to a normalizable \( (\psi)^{-1} \) and the second derivative of \( \ln \psi(x, \varepsilon) \) is well behaved, in a sense to be defined shortly, then neither \( A^{+}(\varepsilon)A^{-}(\varepsilon) \) nor \( A^{-}(\varepsilon)A^{+}(\varepsilon) \) has a normalizable eigenstate with eigenvalue zero. We denote such a solution \( \psi \) by \( \xi \). Therefore \( A^{+}A^{-} \) and \( A^{-}A^{+} \) have identical spectra as depicted in Fig. 1(c). Then

\[ \bar{H} = A^{-}(\varepsilon)A^{+}(\varepsilon) + \varepsilon = H - \frac{d^{2}}{dx^{2}} \ln \xi(x, \varepsilon), \quad (45) \]
has a spectrum identical to that of $H$. The relations between the eigenvalues and the eigenstates are given by

$$E^{(m)} = E^{(m)}, \quad m = 0, 1, 2, \ldots,$$

$$\tilde{\psi}^{(m)} = \exp(i\phi) \left( E^{(m)} - \varepsilon \right)^{-\frac{1}{2}} A^{-} \psi^{(m)},$$

$$\psi^{(m)} = \exp(-i\phi) \left( E^{(m)} - \varepsilon \right)^{-\frac{1}{2}} A^{+} \tilde{\psi}^{(m)},$$

$$A^{\pm}(\varepsilon) = \frac{1}{\sqrt{2}} \left[ \pm \frac{d}{dx} + \frac{d}{dx} \ln \tilde{\zeta}(x, \varepsilon) \right].$$

(46)

The phase factor $\phi$ has been left undetermined. Furthermore, the non-normalizable solutions $\psi$ and $\tilde{\psi}$ for energy $\varepsilon$ are connected by

$$\tilde{\psi}(x, \varepsilon) = \frac{1}{\psi(x, \varepsilon)}. \quad (47)$$

In this section we assume that $-\infty \leq x \leq \infty$ and postpone the discussion of $0 \leq r \leq \infty$ to a later section. It is necessary to make this distinction because the type of singularities of the potential $V$ that are physically admissible depends on the range of values of the variable $x$. Potentials with singularities of the form $r^{-2}$ are admissible for the radial problem, but singularities of the form $x^{-2}$ are inadmissible when $-\infty \leq x \leq \infty$. The discussion of the construction of a normalizable $\psi^{-1}$ depends on the spatial domain in which $\psi$ and $V$ are defined. We now examine the question of the normalizability of $\psi^{-1}$ when $-\infty \leq x \leq \infty$. Let $\phi_1(x, \varepsilon)$ be a nodeless solution of $H\psi = \varepsilon\psi$ for $\varepsilon < E^{(0)}$. The existence of such a solution can be rigorously proved \[21\]. Another linearly independent solution at the same energy is given by

$$\phi_2(x, \varepsilon) = \phi_1(x, \varepsilon)\int_{-\infty}^{x} \frac{dz}{\phi_1^2(z)}. \quad (48)$$

The nodelessness of $\phi_1$ guarantees that this integral is well defined. The general solution at the energy $\varepsilon$ is given by

$$\psi(x, \varepsilon, \alpha) = \phi_1 + \alpha \phi_2 = \phi_1 \left( 1 + \alpha \int_{-\infty}^{x} \frac{dz}{\phi_1^2(z)} \right), \quad (49)$$

in which the lower limit of the integral has been chosen to be $-\infty$ and $\alpha$ is an arbitrary constant. Let

$$\beta(\varepsilon) = \left( \int_{-\infty}^{\infty} \frac{dz}{\phi_1^2(z, \varepsilon)} \right)^{-1}. \quad (50)$$

It can be shown that for values of $\alpha$ in the range $-\beta < \alpha < \infty$, $\psi$ will remain nodeless and $\psi$ is unnormalizable. It can also be shown that for $-\beta < \alpha < \infty$, $\psi^{-1}$ is singularity free and normalizable. This range of values of $\alpha$ then leads to a normalizable $\psi^{-1}$ and $\tilde{\psi} = \psi^{-1}$ corresponds to an eigenstate of $\tilde{H}$ as defined in Eq. (42) with ground state
eigenvalue $E^{(0)} = \varepsilon$. It can be shown that for the limiting values $\alpha = -\beta$ and $\alpha = \infty$ even though $\psi^{-1}$ is unnormalizable the second derivative of $\ln \psi$ is divergence free and finite in the asymptotic region. These values of $\alpha$ then lead to $\tilde{H}$ defined in Eq. (45) with the same spectrum as $H$. But if $\alpha < -\beta$ then $\psi$ will vanish for some finite value of $x$ as can be seen from Eq. (49) and the second derivative of $\ln \psi(x, \varepsilon, \alpha)$ then diverges when $\psi$ vanishes. Hence for $\alpha < -\beta$, $\psi(x, \varepsilon, \alpha)$ does not lead to a physically acceptable potential $\tilde{V}$. The above analysis will be illustrated with examples in the next section.

1.7. Free particle, addition of bound state

Let $V(x) = 0$. $H$ has only a positive energy spectrum. For negative energies $\varepsilon = -\gamma^2/2$, the general solution of $H \psi = \varepsilon \psi$ is given by

$$\psi(x, \varepsilon) = \cosh \gamma x + \alpha \sinh \gamma x . \quad (51)$$

Though $\psi$ is unnormalizable, for values of the parameter $\alpha$ in the range $|\alpha| < 1$ $\psi$ is nodeless and $\psi^{-1}$ is normalizable. The family of potentials

$$\tilde{V} = V - \frac{d^2}{dx^2} \ln \psi(x, \varepsilon) = -\frac{\gamma^2 (1 - \alpha^2)}{(\cosh \gamma x + \alpha \sinh \gamma x)^2} , \quad |\alpha| < 1 , \quad (52)$$

therefore have a single bound state at energy

$$\tilde{E}^{(0)} = -\frac{1}{2} \gamma^2 , \quad (53)$$

with ground state eigenfunction

$$\tilde{\psi}^{(0)} \sim \frac{1}{\psi} = \frac{1}{\cosh \gamma x + \alpha \sinh \gamma x} , \quad |\alpha| < 1 . \quad (54)$$

For positive energies, Eq. (46) then gives

$$\tilde{\psi}(x, \varepsilon) = -\left[2(\tilde{E}^{(0)} - E)\right]^{-\frac{1}{2}} \left(-\frac{d}{dx} + \gamma \frac{\sinh \gamma x + \alpha \cosh \gamma x}{\cosh \gamma x + \alpha \sinh \gamma x}\right) \psi(x, E) . \quad (55)$$

In the asymptotic region $|x| \to \infty$, this equation becomes

$$\lim_{|x| \to \infty} \tilde{\psi}(x, E) = -\left[2(\tilde{E}^{(0)} - E)\right]^{-\frac{1}{2}} \left(-\frac{d}{dx} + \gamma\right) \lim_{|x| \to \infty} \psi(x, E) . \quad (56)$$

The $\alpha$ independence of this equation means that the transmission coefficient of this family of potentials $V(x, E, \alpha)$ are identical. This family of potentials is an example of the phase-equivalent family of Bargmann [22].
1.8. Simple harmonic oscillator, addition of bound state

The oscillator potential does not belong to the category of potentials that remain finite in the asymptotic region. Nevertheless, the oscillator example serves to clarify some of the discussion in the text. The harmonic oscillator Hamiltonian

\[ H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2, \]  

has the eigenvalue spectrum

\[ E = (n + \frac{1}{2}), \quad n = 0, 1, 2, \ldots \]  

(58)

The even solution of \( H \psi = \epsilon \psi \) for all energies can be written in series form [23] and is given by

\[ \phi_1(x, \epsilon) = \left(1 + \frac{\epsilon x^2}{2!} + \delta (4 + \delta) \frac{x^4}{4!} + \delta (4 + \delta)(8 + \delta) \frac{x^6}{6!} + \ldots\right) e^{-x^2/2}, \]  

(59)

where

\[ \delta = 1 - 2\epsilon. \]  

(60)

For energies below the ground state of the oscillator \( \epsilon < E^{(0)} = \frac{1}{2} \), \( \delta > 0 \), which guarantees that \( \phi_1 \) is positive definite. Thus \( \phi_1 \) is a nodeless unnormalizable solution for \( \epsilon < \frac{1}{2} \). The linearly independent solution

\[ \phi_2(x, \epsilon) = \phi_1 \int_0^x \frac{dz}{\phi_1^2(z)}, \]  

(61)

can also be written in series form as

\[ \phi_2(x, \epsilon) = \left(1 + (\delta + 2) \frac{x^2}{3!} + (\delta + 2)(\delta + 6) \frac{x^4}{5!} + \ldots\right) x e^{-x^2/2}. \]  

(62)

\( \phi_2 \) vanishes at \( x = 0 \) but the series within the parentheses is positive definite when \( \epsilon < \frac{1}{2} \). Both \( \phi_1 \) and \( \phi_2 \) may be expressed in terms of standard parabolic cylinder functions. The general solution at energy \( \epsilon \) is then given by

\[ \psi(x, \epsilon) = \phi_1(x, \epsilon) + \alpha \phi_2(x, \epsilon). \]  

(63)

In terms of the parameter

\[ \lim_{x \to \infty} \frac{\phi_1}{\phi_2} = \beta(\epsilon) = \left(\int_0^\infty \frac{dz}{\phi_1^2(z, \epsilon)} \right)^{-1} = 2 \frac{\Gamma\left(\frac{3}{4} - \frac{1}{2} \epsilon\right)}{\Gamma\left(\frac{1}{4} - \frac{1}{2} \epsilon\right)}, \]  

(64)

which arises from asymptotic formulae for the parabolic cylinder functions [23], for \( |\alpha| < \beta \)

\[ \psi(x, \epsilon, \alpha) = \phi_1(x, \epsilon) \left(1 + \alpha \int_0^x \frac{dz}{\phi_1^2(z, \epsilon)} \right), \]  

(65)
The phase-equivalent potentials $\tilde{V}(x, \varepsilon, \alpha)$ are shown for $\varepsilon = -1/2$. The value of $\alpha$ is indicated below each curve. The harmonic oscillator potential $V(x) = x^2/2$ is also shown as a broken curve. The potentials shown in figure by full curves have identical spectra with ground state at $\tilde{E}^{(0)} = -1/2$ as indicated by the horizontal broken line. The rest of the spectrum is identical to that of the harmonic oscillator.

is nodeless and $\psi^{-1}$ is normalizable. The family of Hamiltonians

$$\tilde{H} = H - \frac{d^2}{dx^2} \ln \psi(x, \varepsilon, \alpha), \quad |\alpha| < \beta,$$

therefore have identical spectra given by

$$\tilde{E}^{(0)} = \varepsilon < \frac{1}{2}, \quad \tilde{E}^{(m)} = m - \frac{1}{2}, \quad m = 1, 2, \ldots .$$

(67)

This family of potentials is another example of the phase-equivalent family of Bargmann [22]. Since the energy $\tilde{E}^{(0)}$ is arbitrary as long as $\tilde{E}^{(0)} < \frac{1}{2}$, the above equations give a recipe for constructing anharmonic potentials with spectra defined by Eq. (67).

Using the series expansion for $\phi_1$ and $\phi_2$ the potentials $\tilde{V}(x, \varepsilon, \alpha)$ have been calculated for a range of values of $\varepsilon$ and $\alpha < \beta(\varepsilon)$. Fig. 3 shows $\tilde{V}$ for $\varepsilon = -1/2$ and $\alpha$ in the range of values $0 < \alpha < 2/\sqrt{\pi}$. The results for positive values of $\alpha$ are shown. The potential for the corresponding negative value of $\alpha$ may be obtained by mirror reflection about the $x$-axis. For $\alpha = 0$, $\tilde{V}(x) = x^2/2 - 1$ is a shifted oscillator. This is the only value of $\alpha$ for which $\tilde{V}$ is invariant under the parity transformation. Thus by imposing a specific
The potential $\tilde{V}(x, \varepsilon, \alpha)$ for $\alpha = 0$ and (a) $\varepsilon = 0.45$, (b) $\varepsilon = 0.0$. The harmonic oscillator potential is indicated as a broken curve. The ground state of $V$ at $E = 0.5$ is indicated by a full horizontal line and the ground state of $\tilde{V}$ at $E = \varepsilon$ is indicated by a broken horizontal line. The rest of the spectrum of $\tilde{V}$ is identical to that of the harmonic oscillator.

condition on $\tilde{V}(x)$ a unique member of the family is obtained. Fig. 4 shows $\tilde{V}(x, \varepsilon, 0)$ for a range of values of $\varepsilon$ for a fixed value of $\alpha = 0$. These figures show that for $0 < \varepsilon < \frac{1}{2}$ the ground state of the new potential $\tilde{V}$ lies inside the double well. This is an example of the general result that when $\varepsilon$ lies below the ground state of a given potential $V$, but $\varepsilon > V_{\text{min}}(x)$ where $V_{\text{min}}$ is the minimum of the potential, the resulting partner potential $\tilde{V}(x)$ is necessarily a double well. It can be shown on general grounds that a double well is necessary to accommodate the new level at $\varepsilon$ close to the first excited state of $\tilde{V}(x)$ at energy $E^{(0)}$. Fig. 4 shows double well potentials whose spectrum is fixed by construction to be of the form given in Eq. (67).

We next consider the limiting values $\alpha = \beta(\varepsilon)$ for which $\psi(x, \varepsilon, \beta)^{-1}$ is unnormalizable. The value of $\beta$ for a given value of $\varepsilon$ may be found from Eq. (64). It can be shown that the second derivative of $\ln \psi(x)$ is divergence free. Hence

$$\tilde{V}(\varepsilon) = \frac{x^2}{2} - \frac{d^2}{dx^2} \ln \psi(x, \varepsilon, \pm \beta(\varepsilon)),$$

has the spectrum

$$E^{(m)} = m + \frac{1}{2}, \quad m = 0, 1, 2, \ldots,$$

which is identical to the spectrum of the harmonic oscillator. The eigenfunctions of $\tilde{H}$ may be given in terms of the oscillator eigenfunctions using the intertwining relations involving the $A^\pm$ operators using the logarithmic derivative of $\psi(x, \varepsilon, \beta(\varepsilon))$. Thus Hamiltonians $\tilde{H}(\varepsilon)$ for various values of $\varepsilon < \frac{1}{2}$ which have spectra identical to the harmonic oscillator may be constructed. The potentials $\tilde{V}(x, \varepsilon, \beta(\varepsilon))$ have been calculated.
FIGURE 5. The potential $\tilde{V}(x, \varepsilon, \beta(\varepsilon))$ for a range of values of $\varepsilon$. The $\varepsilon$ value is indicated on each curve. The harmonic oscillator potential is shown as a broken curve. All the potentials shown in this diagram have spectra identical to that of the harmonic oscillator. The asymptotic values of the full curves are given by $\lim_{x \to \pm \infty} \Delta V = 0$, where $\Delta V = \tilde{V}(x) - V(x)$.

numerically and are shown in Fig. 5 for a range of values of $\varepsilon$ and positive values of $\beta(\varepsilon)$. The potentials for negative $\beta$ may be obtained by mirror reflection.

1.9. Summary

It has been demonstrated that the algebra of supersymmetry can be used to find partner Hamiltonians to one-dimensional Hamiltonians. The flexibility in the choice of the partner Hamiltonian enables the identification of different types of supersymmetric pairings. A procedure for constructing Hamiltonians with either identical spectra or with identical spectra apart from a missing ground state has been given. This procedure has been illustrated with several examples. This recipe can be used to either add a new ground state eigenvalue to, or eliminate the ground state of, or maintain the same spectrum of a given Hamiltonian corresponding to a Schrödinger equation. This procedure may be repeated again and again in a suitable combination to generate hierarchies of Hamiltonians whose spectra are related to each other. By applying this procedure to the harmonic oscillator
anharmonic potentials whose spectra are identical to that of the harmonic oscillator or contain a ground state lower in eigenvalue than the ground state of the harmonic oscillator have been constructed.

2. SUSYQM AND INVERSE SCATTERING THEORY

The radial Schrödinger equation corresponding to a definite partial wave is studied. The procedures for finding a new potential by eliminating the ground state of a given potential by adding a bound state below the ground state of a given potential and by generating the phase-equivalent family of a given potential using the supersymmetric pairing of the spectra of the operators $A^+A^-$ and $A^-A^+$ are examined. Four different types of transformations generated by the concept of a supersymmetric partner to a given radial Schrödinger equation are identified and the modifications of the Jost functions for the four transformations are classified. It is argued that the Bargmann class of potentials may be generated using suitable combinations of the four types of transformations.

2.1. Introduction

In the first lecture (§1) it was shown that by using the idea of a supersymmetric partner to a Hamiltonian function $H$ of a single variable it is possible to find another Hamiltonian $\tilde{H}$ which had one of the following features: either (i) the complete spectrum of $\tilde{H}$ is made up of all the eigenvalues of $H$ except the ground state of $H$, or (ii) the complete spectrum of $\tilde{H}$ is made up of all the eigenvalues of $H$ and in addition one further eigenvalue which lies below the ground state of $H$, or (iii) the spectrum of $\tilde{H}$ is identical to that of $H$. It was shown that in all three cases the eigenfunctions of $H$ and $\tilde{H}$ for the common eigenvalues are connected by a linear differential operator. By repeated application of this procedure of either deleting an eigenvalue or adding an eigenvalue or maintaining the same eigenvalues it is possible to generate Hamiltonians whose spectra bear a definite relationship to each other. The inverse scattering theory can also accomplish the same tasks through solving either the Gelfand-Levitan [24] or the Marchenko [25] equations [13, 26, 27]. The aim of this lecture is to elucidate the relationship between the two approaches [28]. It will also be shown that the Bargmann class of potentials [22] may be generated by the application of the concept of supersymmetric pairing.

The radial Schrödinger equation differs from the Schrödinger equation in the space $[−\infty, \infty]$ in essential respects. The boundary conditions on the eigenfunctions and the allowed singularities of the potential are different in the two spaces $[−\infty, \infty]$ and $[0, \infty]$. In this lecture the modifications introduced by switching from $x$ to $r$ will be considered first and then the modifications of the Jost function corresponding to four different types of transformations will be studied.
2.2. The radial Schrödinger equation

We now consider the radial Schrödinger equation for a definite partial wave with the Hamiltonian

\[
H = -\frac{1}{2} \frac{d^2}{dr^2} + V(r),
\]

\[
V(r) = \frac{l(l+1)}{2r^2} + v(r).
\]

The potential \(V(r)\) is assumed to be regular, not singular. Specifically, the potentials discussed here are restricted to be no more singular than \(r^{-2}\) at the origin and decreasing at least as fast as \(r^{-2}\) as \(r \to \infty\).

In this lecture the term ‘normalization constant of the eigenfunction’ will be used often. This term has a specific meaning in the terminology of the inverse scattering theory. All bound state eigenfunctions are understood to be normalized to unity in the usual way to reflect the condition that the total probability of finding the bound particle somewhere in space should be unity. However, in the inverse scattering method the term ‘normalization constant’ is used in a specific sense. The regular solution \(\phi\) of the radial Schrödinger equation is defined to be a solution that satisfies the boundary condition

\[
\lim_{r \to 0} \phi(r,E,l) = \frac{l^{l+1}}{(2l+1)!!}.
\]

The regular solution will grow exponentially as \(r \to \infty\) when \(E\) is not one of the eigenenergies. However, when \(E\) is one of the eigenenergies \(E^{(i)}\) the bound state eigenfunction, which decreases exponentially as \(r \to \infty\), is proportional to the regular solution

\[
\psi(r,E^{(i)},l) = \alpha \phi(r,E^{(i)},l), \quad \int_0^\infty \psi^2 dr = 1.
\]

It is this proportionality constant \(\alpha\) that corresponds to the ‘normalization constant’ referred to in the inverse scattering method. Throughout this lecture the term ‘normalization constant’ will be used in the sense in which it is used in the inverse scattering theory. The term ‘normalizable’ will, however, be used in the usual sense of the word, \(i.e., \int_0^\infty \psi^2 dr\) is finite. In view of the different types of transformations of the radial equation that will be discussed the following notations will be adopted. The eigenfunctions of \(H\) defined in Eq. (70) are denoted by \(\psi^{(i)}\) for the discrete states at energies \(E^{(i)}\), the phase shifts for the continuum states \(\psi(r,E)\) for positive energies \(E = \frac{1}{2}k^2\) are denoted by \(\delta(l,k)\) and the Jost function by \(F(l,k)\). The potentials, eigenstates, phase shifts and Jost function after the supersymmetric transformation are denoted by adding a tilde, \(\tilde{\psi}(r,E)\), for example. The different types of transformations are distinguished by adding a suffix, \(\psi_1(r,E)\), for example. We adopt the notation that \(\psi^{(m)}(r)\) is an abbreviation for \(\psi(r,E^{(m)})\).
2.3. Jost function

In scattering theory the S-matrix may be constructed from the Jost function [29]. The integral representation of the Jost function for a potential \( v(r) \) with \( N \) bound states at energies \( E = E^{(i)} \) and phase shifts \( \delta(l, k) \) at energies \( E = \frac{1}{2} k^2 \) for angular momentum \( l \) is given by (see Chadan and Sabatier [30], for example)

\[
F(l, k) = \prod_{i=1}^{N} \left( 1 - \frac{E^{(i)}}{E} \right) \exp \left( -\frac{2}{\pi} \int_{0}^{\infty} \frac{\delta(l, p) p dp}{p^2 - k^2} \right). \tag{73}
\]

The phase of the Jost function is \(-\delta(l, k)\) while the modulus is given by

\[
|F(l, k)| = \prod_{i=1}^{N} \left( 1 - \frac{E^{(i)}}{E} \right) \exp \left( -\frac{2}{\pi} P \int_{0}^{\infty} \frac{\delta(l, p) p dp}{p^2 - k^2} \right), \tag{74}
\]

where the symbol \( P \) stands for principal value. The spectral density for positive energies is given by

\[
\frac{dP(E)}{dE} = \frac{E(l + \frac{1}{2})}{\pi} |F(l, k)|^{-2}. \tag{75}
\]

Knowledge of the phase shifts for all positive energies, the bound state energies \( E^{(i)} \) and the normalization constants \( C^{(i)} \) associated with each of the bound states enables the complete determination of the potential \( V(r) \).

2.4. Elimination of the ground state

By the methods discussed in §1 it can be shown that \( H \) defined by Eq. (70) has a supersymmetric partner

\[
\tilde{H}_1 = H - \frac{d^2}{dr^2} \ln \psi^{(0)}(r). \tag{76}
\]

Since

\[
\lim_{r \to 0} \psi^{(0)}(r) \sim r^{l+1}, \tag{77}
\]

\( \tilde{H}_1 \) corresponds to the potential

\[
\tilde{V}_1(r) = \frac{(l + 1)(l + 2)}{2r^2} + v(r) - \frac{d^2}{dr^2} \ln \left( \frac{\psi^{(0)}(r)}{r^{l+1}} \right), \tag{78}
\]

where the singularity at the origin has been separated to show the behaviour near the origin. It can be established that

\[
\lim_{r \to 0} \tilde{V}_1(r) = \frac{(l + 1)(l + 2)}{2r^2},
\]

\[
\lim_{r \to \infty} \tilde{V}_1(r) = \frac{l(l + 1)}{2r^2}. \tag{79}
\]
\( \tilde{V}_1 \) has no normalizable state with eigenvalue \( E^{(0)} \) and therefore the ground state of \( V \) is missing from the spectrum of eigen values for \( \tilde{V}_1 \). All the other eigenvalues of the two potentials are the same. The eigenfunctions are related by

\[
\tilde{\psi}^{(m)}_1 = (E^{(m)} - E^{(0)})^{-\frac{1}{2}} A_1^{-1} \psi^{(m)}, \quad m = 1, 2, \ldots ,
\]

\[
A_1^{-1} = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \psi^{(0)}(r) \right].
\]  

(80)

Extension of the above eigenfunction relation to positive energy states and use of the asymptotic forms

\[
\lim_{r \to \infty} \psi(r, E) \sim \sin \left( kr - \frac{1}{2} l \pi + \delta(l, k) \right),
\]

\[
\lim_{r \to \infty} \psi^{(0)}(r) \sim \exp(-\gamma^{(0)} r),
\]

(81)

then gives

\[
\lim_{r \to \infty} \tilde{\psi}_1(r, E) \sim \sin \left( kr - \frac{1}{2} l \pi + \tilde{\delta}_1(l, k) \right),
\]

(82)

where

\[
\tilde{\delta}_1(l, k) = \delta(l, k) - \frac{\pi}{2} - \tan^{-1} \left( \frac{\gamma^{(0)}}{k} \right), \quad E = \frac{1}{2} k^2, \quad E^{(0)} = \frac{1}{2} \left( \gamma^{(0)} \right)^2.
\]  

(83)

The phase shift relation is consistent with the observation that when \( r \to 0 \) the singularity of the potential \( \tilde{V}_1 \) corresponds to \( l \to l + 1 \) which implies an increased repulsion and therefore the phase shift should decrease. In the limit \( k \to 0 \) the phase shift decreases by \( \pi \) which is the correct limit when a bound state has been eliminated. In the limit \( k \to \infty \) the phase shift decreases by \( \pi/2 \). Eqs. (73), (74), (80) and (83) enable the establishment of a relationship between the Jost functions for the potentials \( \tilde{V}_1 \) and \( V \). The phase shift relation given in Eq.(83) may be used to compare the Jost functions for the two potentials. Using the integral relation [31]

\[
\frac{4}{\pi} P \int_0^\infty \frac{\cot^{-1}(p/\gamma)}{p^2 - k^2} - pdp = \ln \left( 1 + \frac{\gamma^2}{k^2} \right),
\]

(84)

we can establish that the Jost functions for the two potentials are related by

\[
\tilde{F}_1(l, k) = \frac{k}{F(l, k)}.
\]

(85)

2.5. Addition of a bound state

The potential \( \tilde{V}_2 \) with a ground state at \( \tilde{E} = -\frac{1}{3} \gamma^2 < E^{(0)} \), i.e., below the ground state of \( V \) in addition to sharing all the eigenvalues of \( V \) can be constructed by the methods
of §1. Since the potential in the radial equation can have singularities of the form $r^{-2}$ the equations of §1 must be recast in an appropriate form. The regular solution in the potential $V$ at energy $E$ denoted by $\phi$ satisfies

$$\lim_{r \to 0} \phi = \frac{r^{l+1}}{(2l+1)!!} , \quad \lim_{r \to \infty} \phi \sim \exp(\tilde{\gamma}r) .$$

(86)

Since the energy $E$ is below the ground state of $V$, $\phi$ is nodeless for $r > 0$ and may be chosen to be positive definite for $r > 0$. The linearly independent solution can be taken to be

$$\eta(r) = \phi(r) \int_r^\infty \frac{dz}{\phi^2(z)} .$$

(87)

It is easy to show that

$$\lim_{r \to 0} \eta(r) \sim r^{-l} , \quad \lim_{r \to \infty} \eta(r) \sim \exp(-\tilde{\gamma}r) .$$

(88)

The function $\eta$ is one of the Jost solutions (see [29], for example) defined by a boundary condition in the asymptotic region. For energies $\tilde{E} < E^{(0)}$, $\eta$ is also a nodeless function and is positive definite. When $\tilde{E}$ is not only less than $E^{(0)}$ but also less than the absolute minimum of the potential $V$ then the positivity of $(V - \tilde{E})$ guarantees that $\phi$ and $\eta$ are monotonically growing functions of $r$ in the directions $r = 0 \to \infty$ and $r = \infty \to 0$, respectively. When $V_{\text{min}} < \tilde{E} < E^{(0)}$, $\phi$ and $\eta$ are no longer monotonically growing functions but nevertheless remain nodeless. These assertions on the behaviour of $\phi$ and $\eta$ may be rigorously proved. The function

$$\psi = \phi \cos \alpha + \eta \sin \alpha ,$$

(89)

is also a nodeless function when $0 < \alpha < \frac{1}{2} \pi$ and $\psi^{-1}$ is a normalizable function for this range of values of $\alpha$ since

$$\lim_{r \to 0} \frac{1}{\psi} = \lim_{r \to 0} \frac{1}{\eta \sin \alpha} \sim r^l ,$$

(90)

and

$$\lim_{r \to \infty} \frac{1}{\psi} = \lim_{r \to \infty} \frac{1}{\phi \cos \alpha} \sim \exp(-\tilde{\gamma}r) .$$

(91)

By the methods of §1 it is easy to infer that when $0 < \alpha < \frac{1}{2} \pi$

$$\tilde{V}_2(r) = V(r) - \frac{d^2}{dr^2} \ln \psi(r, \tilde{E}, \alpha) ,$$

(92)

with

$$\psi(r, \tilde{E}, \alpha) = \phi(r, \tilde{E}) \left( \cos \alpha + \sin \alpha \int_r^\infty \frac{dz}{\phi^2(z, \tilde{E})} \right) ,$$

(93)

has a ground state at

$$\tilde{E}_2^{(0)} = \tilde{E} ,$$

(94)
with eigenfunction
\[ \psi_2^{(0)}(r, \tilde{E}, \alpha) = \frac{1}{\psi(r, \tilde{E}, \alpha)}. \]  \(95\)

All the other eigenvalues of \( \tilde{V}_2 \) are identical to the eigenvalues of \( V \). The other eigenfunctions of \( \tilde{V}_2 \) are given by
\[ \psi_2^{(m)} = -(E^{(m)} - \tilde{E})^{-\frac{1}{2}} A_2^{-1} \psi^{(m)}, \quad m = 1, 2, \ldots, \]
\[ A_2(\tilde{E}, \alpha) = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \psi(r, \tilde{E}, \alpha) \right]. \]  \(96\)

The potential \( \tilde{V}_2 \) may be written in the form
\[ \tilde{V}_2(r, \tilde{E}, \alpha) = \frac{l(l-1)}{2r^2} + v(r) - \frac{d^2}{dr^2} \ln(r \left( r \psi(r, \tilde{E}, \alpha) \right)), \]  \(97\)
where the singularity at the origin has been separated to show the behaviour near the origin. It can be established that
\[ \lim_{r \to 0} \tilde{V}_2 = \frac{l(l-1)}{2r^2}, \quad \lim_{r \to \infty} \tilde{V}_2 = \frac{l(l+1)}{2r^2}. \]  \(98\)

Extension of Eq. (96) to positive energy states and use of the asymptotic forms
\[ \lim_{r \to \infty} \psi(r, E) \sim \sin \left( kr - \frac{1}{2} l \pi + \delta(l, k) \right), \]
\[ \lim_{r \to \infty} \psi(r, \tilde{E}, \alpha) \sim \exp(\gamma r), \]  \(99\)
then gives
\[ \lim_{r \to \infty} \psi(r, \tilde{E}, \alpha) \sim \sin \left( kr - \frac{1}{2} l \pi + \tilde{\delta}(l, k) \right), \]  \(100\)
where
\[ \tilde{\delta}(l, k) = \delta(l, k) + \frac{\pi}{2} + \tan^{-1} \left( \frac{\gamma}{k} \right). \]  \(101\)

The phase shift relation is consistent with the observation that when \( r \to 0 \) the singularity of the potential \( \tilde{V}_2 \) corresponds to \( l \to l - 1 \), which implies a decrease in the repulsion and therefore the phase shift should increase. In the limit \( k \to 0 \) the phase shift increases by \( \pi \) which is the correct limit when the number of bound states increases by one. In the limit \( k \to \infty \) the phase shift increases by \( \pi/2 \). The equations given above show that all members of the family \( \tilde{V}_2(r, \tilde{E}, \alpha) \) lead to identical phase shifts for the same energy \( E \) when \( 0 < \alpha < \frac{1}{2} \pi \). Furthermore, since
\[ \lim_{r \to 0} \frac{d}{dr} \ln \psi(r, \tilde{E}, \alpha) = -\frac{l}{r}, \]  \(102\)
Eq. (96) shows that for a fixed principal quantum number \( m \), \( \lim_{r \to 0} \tilde{\psi}(r, \tilde{E}, \alpha) \) is independent of \( \alpha \) and therefore the excited states of \( \tilde{V}_2(r, \tilde{E}, \alpha) \) for various \( \alpha \) have identical normalizations. However, the normalized ground state eigenfunction

\[
\psi_2^{(0)}(r, \tilde{E}, \alpha) = \frac{(\sin \alpha \cos \alpha)^{1/2}}{\phi (\cos \alpha + \sin \alpha \int_r^{\infty} dz/\phi^2(z))},
\]

\[
\int_0^{\infty} [\tilde{\psi}_2^{(0)}]^2 dr = 1,
\]

shows that

\[
\lim_{r \to 0} \tilde{\psi}_2^{(0)}(r, \tilde{E}, \alpha) \sim \left( \frac{\sin \alpha}{\cos \alpha} \right)^{1/2} r^l.
\]

Hence the ground state eigenfunction of the family of potentials \( \tilde{V}_2(r, \tilde{E}, \alpha) \) have different normalizations, i.e., different proportionality to the regular solution, although they belong to the same eigenvalue. It has been shown that the phase shifts, the eigenvalues and the normalization constants of the excited states are identical for all members of the family of potentials \( \tilde{V}_2(r, \tilde{E}, \alpha) \), \( 0 < \alpha < \frac{1}{2} \pi \), while the ground state eigenfunctions belonging to the eigenvalue \( \tilde{E}_2^{(0)} \) = \( \tilde{E} \) have different normalization constants for different values of \( \alpha \). Clearly the family \( \tilde{V}_2(r, \tilde{E}, \alpha) \) is an example of the phase equivalent family which was discussed by Bargmann [22].

The phase shifts and the bound state energies of \( V \) and \( \tilde{V}_2 \) enable the comparison of the Jost functions. From Eqs. (73) and (102) it is easy to show that

\[
\frac{\tilde{F}_2(l, k)}{F(l, k)} = \frac{k - i \tilde{\gamma}}{k}.
\]

### 2.6. Boundary values of \( \alpha \) and equivalent potentials

When the parameter \( \alpha \) lies outside the range \( 0 < \alpha < \frac{1}{2} \pi \), the eigenfunction \( \psi \) in Eq. (89) does not lead to a normalizable \( \psi^{-1} \). When \( -\pi < \alpha < 0 \) or \( \pi > \alpha > \frac{1}{2} \pi \), \( \psi \) vanishes at some finite value of \( r \) because either \( \sin \alpha \) or \( \cos \alpha \) assumes negative values and \( \int_r^{\infty} dz/\phi^2 \) can take all values from 0 to \( \infty \). If \( \psi \) vanishes at a finite value of \( r \) then the second derivative of \( \ln \psi \) diverges at this point. This would then lead to a singular \( V \). However, the critical values \( \alpha = 0 \) and \( \alpha = \frac{1}{2} \pi \) must be studied separately.

(a) When \( \alpha = 0 \)

\[
\lim_{r \to 0} \psi(r, \tilde{E}, 0) = \phi, \quad \lim_{r \to 0} \psi \sim r^{l+1}, \quad \lim_{r \to \infty} \psi = \exp(\tilde{\gamma}r).
\]

The vanishing value of \( \psi \) at \( r = 0 \) shows that \( \psi^{-1} \) is not normalizable, but

\[
\lim_{r \to 0} \frac{d^2}{dr^2} \ln \phi \sim -\frac{(l + 1)}{r^2}, \quad \lim_{r \to \infty} \frac{d^2}{dr^2} \ln \phi \sim 0.
\]
supersymmetric partner to $V$. It can be shown that

$$\tilde{V}_3(r, \tilde{E}) = \frac{(l+1)(l+2)}{2r^2} + \nu(r) - \frac{d^2}{dr^2} \ln \left( \frac{\phi(r, \tilde{E})}{r^{l+1}} \right),$$

and that

$$\lim_{r \to 0} \tilde{V}_3(r, \tilde{E}) = \frac{(l+1)(l+2)}{2r^2}, \quad \lim_{r \to \infty} \tilde{V}_3(r, \tilde{E}) = \frac{l(l+1)}{2r^2}. \quad (109)$$

The eigenvalue spectrum of $\tilde{V}_3$ is identical to that of $V$ and the new eigenfunctions are

$$\tilde{\psi}_3^{(m)}(l, \tilde{E}) = (E^{(m)} - \tilde{E})^{-\frac{1}{2}} A_3 \psi^{(m)},$$

$$A_3^{(l)}(\tilde{E}) = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \phi(r, \tilde{E}) \right]. \quad (110)$$

The phase shifts in the potentials are related by

$$\tilde{\delta}_3(l, k) = \delta(l, k) - \frac{\pi}{2} + \tan^{-1} \left( \frac{\tilde{\gamma}}{k} \right). \quad (111)$$

The phase shift relation is consistent with the observation that when $r \to 0$ the singularity of the potential $\tilde{V}_3$ corresponds to $l \to l + 1$, which implies an increased repulsion and therefore the phase shift should decrease. In the limit $k \to \infty$ the phase shift decreases by $\pi/2$. In the limit $k \to 0$ the phase shift is unchanged which is the correct limit since the two potentials have the same number of bound states. The family of potentials $\tilde{V}_3(r, \tilde{E})$ for different values of $\tilde{E} < E^{(0)}$ have identical spectra but different phase shifts for the same energy. The Jost functions for the potentials $V$ and $\tilde{V}_3$ can be shown to be related in the manner

$$\tilde{F}_3(l, k) = \frac{k}{F(l, k) + i\tilde{\gamma}}. \quad (112)$$

(b) When $\alpha = \frac{1}{2} \pi$

$$\psi(r, \tilde{E}, \frac{1}{2} \pi) = \eta, \quad \lim_{r \to 0} \sim r^{-1}, \quad \lim_{r \to \infty} \psi \sim \exp(-\tilde{\gamma}r). \quad (113)$$

Hence $\alpha = \frac{1}{2} \pi$ does not lead to a normalizable $\psi^{-1}$, because $\psi^{-1}$ diverges as $r \to \infty$. However,

$$\lim_{r \to 0} \frac{d^2}{dr^2} \ln \eta = \frac{l}{r^2}, \quad \lim_{r \to \infty} \frac{d^2}{dr^2} \ln \eta \sim 0. \quad (114)$$

These conditions together with the absence of any singularities in the second derivative of $\ln \eta$ ensure that a singularity-free supersymmetric partner to $V$ may be constructed. It can be shown that

$$V_4(r, \tilde{E}) = \frac{l(l-1)}{2r^2} + \nu(r) - \frac{d^2}{dr^2} \ln \left( r' \eta(r, \tilde{E}) \right), \quad (115)$$
and that
\[
\lim_{r \to 0} \tilde{V}_4(r, \tilde{E}) = \frac{l(l-1)}{2r^2}, \quad \lim_{r \to \infty} \tilde{V}_4(r, \tilde{E}) = \frac{l(l+1)}{2r^2}.
\] (116)

The eigenvalue spectrum of \( \tilde{V}_4 \) is identical to that of \( V \) and the eigenfunctions are related by
\[
\tilde{\Psi}_4^{(m)} = -(E^{(m)} - \tilde{E})^{-\frac{1}{2}} A_4 \psi^{(m)},
\]
\[
A_4(\tilde{E}) = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \eta(r, \tilde{E}) \right].
\] (117)

The phase shifts in the potentials are related by
\[
\tilde{\delta}_4(l, k) = \delta(l, k) + \frac{\pi}{2} - \tan^{-1} \left( \frac{\tilde{\gamma}}{k} \right).
\] (118)

The phase shift relation is consistent with the observation that when \( r \to 0 \) the singularity of the potential \( \tilde{V}_4 \) corresponds to \( l \to l - 1 \), which implies a decreased repulsion and therefore the phase shift should increase. In the limit \( k \to \infty \) the phase shift increases by \( \pi/2 \). In the limit \( k \to 0 \) the phase shift is unchanged which is the correct limit since the two potentials have the same number of bound states. The family of potentials \( \tilde{V}_4(r, \tilde{E}) \) for different values of \( \tilde{E} < E^{(0)} \) have identical spectra but different phase shifts for the same energy. The Jost functions for the potentials \( V \) and \( \tilde{V}_4 \) can be shown to be related in the manner
\[
\tilde{F}_4(l, k) = \frac{k + i \tilde{\gamma}}{k}.
\] (119)

2.7. Summary

It has been shown that by a suitable factorization of the radial Schrödinger equation, it is possible to discover an underlying supersymmetric algebra. This algebra may be used to generate four different types of transformations of the radial Schrödinger equation. The four transformations may be classified as follows.

1. \( T_1 \) is a transformation that eliminates the ground state \([E^{(0)}, \psi^{(0)}]\) of the potential \( V(r) \), leaves the rest of the spectrum of eigenvalues unaltered, leaves the angular momentum component of the potential in the region \( r \to \infty \) unaltered and alters the singular behaviour near \( r \to 0 \) to a centrifugal potential corresponding to angular momentum \((l + 1)\). \( T_1 \) also changes the Jost function corresponding to \( V \) by a multiplicative factor \( k/(k - i\tilde{\gamma}^{(0)}) \) where \( \tilde{\gamma}^{(0)} = [-2E^{(0)}]^{\frac{1}{2}} \). The new eigenfunctions in the potential (78) are given by Eq. (80).

2. \( T_2 \) is a transformation that adds a bound state \( \tilde{E}_2^{(0)}, \tilde{\Psi}_2^{(0)} \) below the ground state of the potential \( V(r) \), leaves the rest of the spectrum of eigenvalues unaltered, leaves the angular momentum component of the potential in the region \( r \to \infty \) unaltered and alters the singular behaviour near \( r \to 0 \) to a centrifugal potential corresponding to angular momentum \((l - 1)\). \( T_2 \) also changes the Jost function corresponding to \( V \) by
a multiplicative factor \((k - i\tilde{\gamma}^{(0)})/k\) where \(\tilde{\gamma}^{(0)} = [-2\tilde{E}^{(0)}]^{1/2}\). The new eigenfunctions in the potential in Eqs. (92) and (93) are given by Eqs. (95) and (96).

(3) \(T_3\) is a transformation that maintains the eigenvalue spectrum of \(V\) unaltered, leaves the angular momentum component of the potential in the region \(r \to \infty\) unaltered and alters the singular behaviour near \(r \to 0\) to a centrifugal potential corresponding to angular momentum \((l+1)\). \(T_3\) also alters the Jost function for \(V\) by a multiplicative factor \(k/(k+i\tilde{\gamma})\) where \(\tilde{\gamma} = [-2\tilde{E}]^{1/2}\) and \(\tilde{E} < E^{(0)}\). The new eigenfunctions in the potential (108) are given by Eq. (110).

(4) \(T_4\) is a transformation that maintains the eigenvalue spectrum of \(V\) unaltered, leaves the angular momentum component of the potential in the region \(r \to \infty\) unaltered and alters the singular behaviour near \(r \to 0\) to a centrifugal potential corresponding to angular momentum \((l-1)\). \(T_4\) also alters the Jost function for \(V\) by a multiplicative factor \((k+i\tilde{\gamma})/k\) where \(\tilde{\gamma} = [-2\tilde{E}]^{1/2}\) and \(\tilde{E} < E^{(0)}\). The new eigenfunctions in the potential (115) are given by Eq. (117).

In the one-dimensional case \(-\infty < x < \infty\), the singularities in the potential at \(x = 0\) are not permitted and the physical eigenfunctions are defined by boundary conditions at \(\pm \infty\). In the case of the radial equation the boundary conditions on the eigenfunctions at \(r = 0\) and \(r = \infty\) are different. All four types of transformations listed above have analogues in the space \([-\infty, \infty]\) but with the difference that no singularities of the type \(x^{-2}\) should be introduced by the transformations because of the boundary conditions usually imposed on \(\psi(x)\). In the \(x\) space \(\tilde{V}_3\) and \(\tilde{V}_4\) arise from considering the limiting values \(\alpha = \infty\) and \(\alpha = -\beta\) discussed in §1.6.

### 2.8. Relationship to Bargmann potentials

It has been shown above that each of the transformations \(T_1 - T_4\) corresponds to a multiplication of the Jost function by a specific rational function of \(k\). By repeated application of a combination of the four types of transformations in an appropriate order the Jost function of \(V\) can be modified by any rational function of \(k\). The generation of the Bargmann class of potentials [7] corresponds to such a modification of the Jost function. Therefore it is clear that the Bargmann class of potentials may be generated by a suitable combination of \(T_1, T_2, T_3,\) and \(T_4\). For example, the multiplication of the Jost function by a factor \((k+ib)/(k+ia)\) can be broken down into two steps, multiplication by \((k+ib)/k\) followed by a further multiplication by \(k/(k+ia)\), corresponding to application of \(T_4\) followed by \(T_3\). The physically acceptable Jost functions must satisfy the condition

\[
\lim_{k \to \infty} F(l,k) = 1 ,
\]  

(120)

and the symmetry relation

\[
F(l, -k^*) = [F(l,k)]^* .
\]

(121)

The modifications of \(F\) introduced by \(T_1-T_4\) clearly satisfies these conditions.
3. SUPERSYMMETRY AND THE DIRAC EQUATION FOR A CENTRAL COULOMB FIELD

It is shown that the methods of Supersymmetric Quantum Mechanics can be used to obtain the complete energy spectrum and eigenfunctions of the Dirac equation for an attractive Coulomb potential.

3.1. The Dirac equation for the Coulomb potential

The Dirac equation for the electron in an attractive central Coulomb field leads to the energy eigenvalue spectrum shown schematically in Fig. 6. The conventional spectroscopic classification of the levels in the non-relativistic limit is indicated alongside the levels.

When the spectrum is unscrambled in this fashion, it is clear that the arrangement of the pair of levels for a fixed value of the total spin $J$ resembles a ‘supersymmetric’ pairing in which the ‘fermionic’ ladder has a spectrum identical to the ‘bosonic’ ladder except for the missing ground state. The level scheme in Fig. 6 corresponds to a juxtaposition of one such ‘supersymmetric’ pair of ladders for each possible value of $J$. We now show that the eigenvalue spectrum and the eigenfunctions of the Dirac equation may indeed be obtained using the methods of SUSYQM.

Adopting the notation used in Bjorken and Drell [32] and defining the parameters

$$\gamma = \frac{ze^2}{c \hbar}, \quad \alpha_1 = m + E, \quad \alpha_2 = m - E,$$

(122)

FIGURE 6. Schematic eigenvalue spectrum of the Dirac equation for the central Coulomb potential $V(r) = -\gamma/r$. $J$ is the total spin and $k = \pm (J + 1/2)$. The quantum numbers are explained in the text.
the coupled radial equations satisfied by the two-component eigenfunction \((G_k, F_k)\) may be written in the matrix form
\[
\begin{pmatrix}
\frac{dG_k}{dr} & 0 \\
0 & \frac{dF_k}{dr}
\end{pmatrix}
+ \frac{1}{r}
\begin{pmatrix}
k & -\gamma \\
-\gamma & k
\end{pmatrix}
\begin{pmatrix}
G_k \\
F_k
\end{pmatrix}
= \begin{pmatrix}
0 & \alpha_1 \\
\alpha_2 & 0
\end{pmatrix}
\begin{pmatrix}
G_k \\
F_k
\end{pmatrix},
\tag{123}
\]
in which \(k\) is an eigenvalue of the operator \(-\sigma \cdot L + 1\) with the allowed values \(k = \pm 1, \pm 2, \pm 3, \ldots\) and satisfies \(|k| = J + 1/2\). In the representation in which Eq. (123) is written, \(J\) and \(k\) are good quantum numbers. \(G_k\) is the ‘large’ component in the non-relativistic limit. The radial functions \(G_k\) and \(F_k\) must be multiplied by appropriate two-component angular eigenfunctions to make up the full four-component solutions of the Dirac equation. Fig. 6 shows that when we compare the two ladders of levels for a fixed value of \(J\), a pair of degenerate levels corresponds to the same value of \(J\) but opposite values of \(k\) except for the lowest state of the pair of ladders when only the negative value of \(k\) corresponds to an eigenstate. We now show how this ladder structure may be related to the pairing of states characteristic of supersymmetric theories (Sukumar [33]).

The matrix multiplying \(1/r\) in Eq. (123) may be diagonalized using the matrix \(D\) given by
\[
D = \begin{pmatrix}
k + s & -\gamma \\
-\gamma & k + s
\end{pmatrix}, \quad s = (k^2 - \gamma^2)^{1/2},
\tag{124}
\]
and its inverse. Multiplication of the matrix differential equation from the left by \(D\) and introduction of the new variable \(\rho = Er\) leads to
\[
\begin{align*}
\left(\frac{k + m}{s} + \frac{1}{E}\right) \tilde{F} &= \left(\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right) \tilde{G}, \\
\left(\frac{k - m}{s} + \frac{1}{E}\right) \tilde{G} &= \left(-\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right) \tilde{F}.
\tag{125}
\end{align*}
\]
where
\[
\begin{pmatrix}
\tilde{G} \\
\tilde{F}
\end{pmatrix} = D \begin{pmatrix}
G_k \\
F_k
\end{pmatrix}.
\tag{126}
\]
These equations are similar to the relation between the two components of the eigenfunctions of a supersymmetric Hamiltonian
\[
H = \{Q, Q^\dagger\}, \quad Q = \begin{pmatrix}
0 & 0 \\
A_0^- & 0
\end{pmatrix}, \quad Q^\dagger = \begin{pmatrix}
0 & A_0^+ \\
0 & 0
\end{pmatrix},
\tag{127}
\]
and
\[
A_0^\pm = \left(\pm\frac{d}{d\rho} + \frac{s}{\rho} - \frac{\gamma}{s}\right).
\tag{128}
\]
The nilpotent operator \(Q\) commutes with \(H\) and therefore corresponds to a conserved charge of this system. \(Q\) and \(Q^\dagger\) induce transformations between the ‘bosonic’ sector represented by \(\tilde{F}\) and the ‘fermionic’ sector represented by \(\tilde{G}\). Eq. (125) may be viewed as a representation of such a transformation. In supersymmetric Quantum Mechanics the ‘fermionic’ and ‘bosonic’ components have identical spectra except for the ground state of the ‘bosonic’ sector which is annihilated by the charge operator \(Q\).
In other words the eigenvalue equations for $\tilde{F}$ and $\tilde{G}$

\[
A_0^+ A_0^- F = \left( \frac{\gamma^2}{s^2} + \frac{1 - m^2}{E^2} \right) F,
\]
\[
A_0^- A_0^+ \tilde{G} = \left( \frac{\gamma^2}{s^2} + \frac{1 - m^2}{E^2} \right) \tilde{G},
\]

show that every eigenvalue of $A_0^+ A_0^-$ is also an eigenvalue of $A_0^- A_0^+$ except when $A_0^- \tilde{F} = 0$. The condition $A_0^- \tilde{F} = 0$ leads to the ground state eigenfunction

\[
\tilde{F}^{(0)} = \rho^s \exp(-\gamma \rho / s),
\]

and the ground state energy eigenvalue is given by

\[
\left( E_F^{(0)} \right)^2 = \frac{m^2}{1 + \gamma^2 / s^2}.
\]

$A_0^- A_0^+$ has no normalizable eigenstate at this energy. All the other states of $A_0^+ A_0^-$ and $A_0^- A_0^+$ are paired and the eigenfunctions are linked in the form $\tilde{F} \sim A_0^+ \tilde{G}$ and $\tilde{G} \sim A_0^- \tilde{F}$ as indicated in Eq. (125). We now show how the ground state of $A_0^- A_0^+$ may be obtained.

The uncoupled second order differential equations for $\tilde{F}$ and $\tilde{G}$

\[
\left[ \frac{d^2}{d\rho^2} + \frac{2\gamma}{\rho} \frac{s(s-1)}{\rho^2} + 1 - \frac{m^2}{E^2} \right] \tilde{F} = 0,
\]
\[
\left[ \frac{d^2}{d\rho^2} + \frac{2\gamma}{\rho} \frac{s(s+1)}{\rho^2} + 1 - \frac{m^2}{E^2} \right] \tilde{G} = 0,
\]

show that Eq. (133) may be obtained from Eq. (132) by the replacement $s \to s + 1$. This suggests that Eq. (133) may be written in the form

\[
A_1^+ A_1^- \tilde{G} = \left[ \frac{\gamma^2}{(s+1)^2} + 1 - \frac{m^2}{E^2} \right] \tilde{G},
\]

with

\[
A_1^+ = \left[ \pm \frac{d}{d\rho} + \frac{s+1}{\rho} - \frac{\gamma}{s+1} \right].
\]

$\tilde{G}$ has a supersymmetric partner $\tilde{H}$ which satisfies

\[
A_1^- A_1^+ \tilde{H} = \left[ \frac{\gamma^2}{(s+1)^2} + 1 - \frac{m^2}{E^2} \right] \tilde{H}.
\]

Just as $\tilde{F}$ and $\tilde{G}$ may be viewed as the component eigenfunctions of a supersymmetric Hamiltonian so also $\tilde{G}$ and $\tilde{H}$ may be viewed as the component eigenfunctions of another supersymmetric Hamiltonian. The spectrum of $\tilde{H}$ is identical with that of $\tilde{G}$ except for a missing state at the energy corresponding to the ground state eigenvalue of $A_1^+ A_1^-$. By
the same reasoning as for $A_0^+A_0^-$ we may then infer that the ground state of $A_1^+A_1^-$ has

the eigenfunction

$$\tilde{G}^{(0)} = \exp[-\gamma \rho/(s+1)], \quad (137)$$

with a ground state energy eigenvalue which satisfies

$$\left(\frac{E_G^{(0)}}{m^2}\right)^2 = \frac{m^2}{1+\gamma^2/(s+1)^2}. \quad (138)$$

The excited states of $A_1^+A_1^-$ satisfy equations similar to Eq. (125). Explicitly

$$\left[\frac{k}{s+1} + \frac{m}{E}\right] \tilde{G} = A_1^+\tilde{H},$$

$$\left[\frac{k}{s+1} - \frac{m}{E}\right] \tilde{H} = A_1^-\tilde{G}. \quad (139)$$

Having obtained the ground state $\tilde{G}^{(0)}$ we can now use the supersymmetric pairing of $\tilde{F}$ and $\tilde{G}$ to obtain the first excited state of $A_0^+A_0^-$ in the form

$$\tilde{F}^{(1)} \sim A_0^+\tilde{G}^{(0)}, \quad E_F^{(1)} = E_G^{(0)}, \quad (140)$$

where the suffixes carry obvious meaning.

This procedure may be repeated to find the ground state of a hierarchy of operators $A_2^+A_2^-, A_3^+A_3^-, \ldots$ with each iteration corresponding to a shift of $s$ by 1 in the definition of the $A^\pm$ operators. This hierarchy corresponds to the Hamiltonian hierarchy discussed in §1. From the ground state properties of the members of this hierarchy all the excited state eigenfunctions and eigenvalues of $A_0^+A_0^-$ can be obtained. The allowed energy eigenvalues of Eq. (125) are

$$E^{(n)} = \frac{m}{\sqrt{1+\gamma^2/(s+n)^2}}, \quad n = 0, 1, 2, \ldots, \quad (141)$$

and the eigenfunctions may be written in the form

$$\tilde{F}^{(n)} \sim (A_0^+A_1^+\ldots A_n^+) \rho^{s+n} \exp[-\gamma \rho/(s+n)], \quad (142)$$

with

$$A_n^\pm = \pm \frac{d}{d\rho} + \frac{s+n}{\rho} - \frac{\gamma}{s+n}. \quad (143)$$

Although $\tilde{F}$ and $\tilde{G}$ satisfy uncoupled second order differential equations, the normalization of one of them determines that of the other as is required by Eq. (125). The solution of Eq. (123) can be written, after adding the suffix $n$, in the form

$$\left(\begin{array}{c} G_k^{(n)} \\ F_k^{(n)} \end{array}\right) = \frac{1}{2s(k+s)} \left(\begin{array}{cc} k+s & \gamma \\ \gamma & k+s \end{array}\right) \left(\begin{array}{c} (k/s-m/E)^{-1}A_0^-\tilde{F}^{(n)} \\ \tilde{F}^{(n)} \end{array}\right). \quad (144)$$
When $n = 0$, $\tilde{G}$ has no normalizable eigenstate at energy $E_F^{(0)}$. We treat the two possible values of $k$ for this energy separately. When $k = -sm/E_F^{(0)}$ Eq. (125) requires that we must choose

$$
\tilde{F}(-k, \rho) = \rho^s \exp(-\gamma \rho / s), \quad \tilde{G}(-k, \rho) = 0,
$$

(145)

to obtain normalizable solutions for $F_k$ and $G_k$. However, when $k = +sm/E_F^{(0)}$ Eq. (125) requires that we must choose

$$
\tilde{F}(+k, \rho) = \rho^s \exp(-\gamma \rho / s),
\quad \tilde{G}(+k, \rho) = \rho^{-s} \exp(+\gamma \rho / s) \int^\rho x^{2s} \exp(-2\gamma x / s) dx.
$$

(146)

Since $\tilde{G}$ is not normalizable, the positive value of $k$ does not lead to normalizable solutions for $F_k$ and $G_k$.

The expression for the spectrum, Eq. (141), is an even function of $k$. A fixed $|k|$ leads to a doublet of states corresponding to $k = +|k|$ and $k = -|k|$ degenerate in energy for all positive integral values of $n$. For $n = 0$, only the negative value of $k$ leads to normalizable $F_k$ and $G_k$ and therefore the state with $n = 0$ is a singlet. This explains the ladder structure of the spectrum for a fixed value of $|k| = J + 1/2$ and opposite values of $k$.

The above analysis considered a fixed $k$ with the corresponding $s = (k^2 - \gamma^2)^{1/2}$. However, in Eqs. (141)-(143) $s$ enters only as a parameter. Hence we can obtain the complete spectrum and the eigenfunctions of Eq. (123) for all values of $J = |k| - 1/2$ by the above procedure. The complete spectrum is given by

$$
E_k^{(n)} = \frac{m}{\sqrt{1 + \gamma^2 / \left[n + (k^2 - \gamma^2)^{1/2}\right]^2}}, \quad n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots,
$$

(147)

and Eq. (142) for the eigenfunctions $\tilde{F}^{(n)}$ is valid for different values of $s(k)$. The principal quantum number $N$ is related to $n$ and $k$ by the relation $N = n + |k|$.

We have shown that the supersymmetric pairing of $\tilde{F}$ and $\tilde{G}$ enables an elegant treatment of the Dirac equation for a central Coulomb field (Darwin [34], Biedenharn [35]). The interaction of a charged particle with the vacuum fluctuations of the quantized radiation field leads to departures from the Coulomb potential. When the field deviates from a Coulomb field, the transformed states $\tilde{F}$ and $\tilde{G}$ no longer belong to a supersymmetric pair and the eigenvalue spectrum loses the ladder structure characteristic of supersymmetric pairing. Therefore we may say that the Lamb shift, which is an effect due to vacuum fluctuations, is related to the breaking of the supersymmetry that connects $\tilde{F}$ and $\tilde{G}$. 
4. SUPERSYMMETRY, POTENTIALS WITH BOUND STATES AT ARBITRARY ENERGIES AND MULTI-SOLITON SOLUTIONS OF KDV EQUATION

The connection between the algebra of supersymmetry and the inverse scattering method can be used to construct reflectionless potentials with any specified number of non-degenerate bound states at arbitrary energies. The reflection coefficient of the potential so constructed is related to the reflection coefficient of a potential which supports no bound states. By choosing the reference potential to be $V = 0$ it is possible to construct reflectionless potentials with bound states at arbitrary energies. It is well known that the KdV equation has multi-soliton solutions with known analytical form. The $N$-soliton and the $(N + 1)$-soliton solutions of the KdV equation may be shown to be connected by a transformation identical to that of the transformation linking the eigenfunctions of the bosonic and fermionic sectors of a Supersymmetric pair. The reflectionless potential with $N$ bound states constructed using the ideas of the SUSY method to plant bound states can be shown to be identical to the $N$-soliton solution of the KdV equation for appropriate choice of a set of parameters.

4.1. Potentials with a single bound state

Let $V_0(x)$ be a potential that supports no bound states and $R_0(k)$ be the reflection coefficient for positive energies. Using the procedure outlined in §1.7 ([18]) it is possible to find a potential $V_1$ which supports a single bound state at energy $E_1 = -\gamma_1^2/2$. It was shown in §1.7 that $V_1$ may be written in the form

$$V_1 = V_0 - \frac{d^2}{dx^2} \ln \psi_0(E_1),$$

where $\psi_0(E_1)$ is a nodeless unnormalizable solution of the Schrödinger equation for the potential $V_0$ at energy $E_1$. It was shown in §1.7 that the ground state eigenfunction of $V_1$ at energy $E_1$ is given by

$$\psi_1(E_1) \sim \frac{1}{\psi_0(E_1)},$$

while for $E \neq E_1$

$$\psi_1(E) \sim A_0^-(E_1) \psi_0(E),$$

where

$$A_0^-(E_1) = \frac{1}{\sqrt{2}} \left[ - \frac{d}{dx} + \frac{d}{dx} \ln \psi_0(E_1) \right].$$

The reflection coefficient of $V_1$ is given by

$$R_1(k) = \frac{\gamma_1 - ik}{\gamma_1 + ik} R_0(k).$$
The above results simplify for the case of a free particle for which $V_0 = 0$ and $R_0(k) = 0$. Eq. (152) shows that the reflection coefficient for the supersymmetric partner vanishes identically. The non-normalizable solution in $V_0$ for energy $E_1$ is given by

$$\psi_0(E_1) = \cosh \gamma_1 x + \alpha_1 \sinh \gamma_1 x, \quad |\alpha_1| < 1 .$$

(153)

The condition $|\alpha_1| < 1$ ensures that $\psi_0(E_1)$ is a nodeless function even though it is not normalizable. The reflectionless potential $V_1$ with a single bound state at $E_1$ is given by

$$V_1 = -\gamma_1^2 \text{sech}^2 (\gamma_1 x + \tanh^{-1} \alpha_1) ,$$

(154)

with the ground state eigenfunction

$$\psi_1(E_1) \sim \text{sech} (\gamma_1 x + \tanh^{-1} \alpha_1) .$$

(155)

When $\alpha_1 = 0$, $V_1$ is a symmetric reflectionless potential. Using the suffix SR to denote ‘symmetric and reflectionless’ and using the notation that the normalized eigenfunctions will be denoted by the addition of a tilde, in terms of the normalized ground state eigenfunction

$$\tilde{\psi}_{1SR}(E_1) = (\gamma_1/2)^{1/2} \text{sech} \gamma_1 x ,$$

(156)

the potential $V_{1SR}$ may be written in the form

$$V_{1SR} = -2\gamma_1 \tilde{\psi}_{1SR}^2(E_1) .$$

(157)

### 4.2. Potentials with two bound states

The procedure used in the previous subsection may be repeated to find a potential with two bound states at energies $E_1$ and $E_2$. $V_2$ is given by

$$V_2 = V_1 - \frac{d^2}{dx^2} \ln \psi_1(E_2) ,$$

(158)

where $\psi_1(E_2)$ is the nodeless non-normalizable solution of the Schrödinger equation for the potential $V_1$ at energy $E_2 = -\gamma_2^2/2$. The ground state eigenfunction of $V_2$ is given by

$$\psi_2(E_2) \sim \frac{1}{\psi_1(E_2)} ,$$

(159)

while the eigenfunctions for other energies are given by

$$\psi_2(E) \sim A^-_1(E_2) \psi_1(E) , \quad E \neq E_2 ,$$

(160)

where

$$A^-_1(E_2) = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + \frac{d}{dx} \ln \psi_1(E_2) \right] .$$

(161)
In particular the first excited state of $V_2$ at energy $E_1$ has the eigenfunction
\[ \psi_2(E_1) \sim A_1^{-1}(E_2) \psi_1(E_1) . \] (162)

Using Eqs. (148) and (158) the potential $V_2$ may be written in the form
\[ V_2 = V_0 - \frac{d^2}{dx^2} \ln[\psi_0(E_1) \psi_1(E_2)] . \] (163)

The reflection coefficient of $V_2$ for positive energies is given by
\[ R_2(k) = \frac{(\gamma_2 - ik)(\gamma_1 - ik)}{(\gamma_2 + ik)(\gamma_1 + ik)} R_0(k) . \] (164)

The above expressions for $V_2$ and $\psi_2$ are given in terms of the solution $\psi_1$ in the potential $V_1$. It would be more convenient to express all quantities in terms of the solutions in the reference potential $V_0$ which has no bound states. Eqs. (150) and (151) show that
\[ \psi_1(E_2) \sim \left[ -\frac{d}{dx} + \frac{d}{dx} \ln \psi_0(E_1) \right] \psi_0(E_2) . \] (165)

Hence
\[ \psi_0(E_1) \psi_1(E_2) \sim \det D_2 , \] (166)
where $D_2$ is a matrix given by
\[ D_2 = \begin{pmatrix} \psi_0(E_1) & \psi_0(E_2) \\ \psi_0(E_1) & \psi_0(E_2) \end{pmatrix} . \] (167)

These expressions may be used to write the potential with two bound states in the form
\[ V_2 = V_0 - \frac{d^2}{dx^2} \ln \det D_2 . \] (168)

The ground state eigenfunction is given by
\[ \psi_2(E_2) \sim \frac{\psi_0(E_1)}{\det D_2} \sim \left[D_2^{-1}\right]_{22} . \] (169)

The eigenfunction for the first excited state of $V_2$ may be simplified to the form
\[ \psi_2(E_1) \sim \frac{\psi_0(E_2)}{\det D_2} \sim \left[D_2^{-1}\right]_{12} . \] (170)

Thus the potential $V_2$ is expressed in terms of the second derivative of the determinant of $D_2$ while the eigenfunctions of $V_2$ are given in terms of the elements in the last column of the inverse of the matrix $D_2$. The condition that $\psi_0(E_1)$ and $\psi_1(E_2)$ must be chosen to be nodeless is equivalent to the requirement that $\psi_0(E_1)$ and $\psi_0(E_2)$ must be chosen such that the determinant of $D_2$ is free of zeros.
To illustrate the above results we consider the case of the free particle, $V_0 = 0$. For this case the reflection coefficient for the potential with two bound states vanishes. Since

\[
\psi_0(E_1) = \cosh \gamma_1 x + \alpha_1 \sinh \gamma_1 x , \\
\psi_0(E_2) = \sinh \gamma_2 x + \alpha_2 \cosh \gamma_2 x ,
\]

(171)

the condition that the determinant of $D_2$ be free of zeros can be met only if $|\alpha_1| < 1$ and $|\alpha_2| < 1$. The symmetric reflectionless potential with bound states at $E_1$ and $E_2$, obtained by choosing $\alpha_1 = 0$ and $\alpha_2 = 0$, is given by

\[
V_{2SR} = -\frac{d^2}{dx^2} \ln \det D_{2SR} ,
\]

(172)

where

\[
D_{2SR} = \begin{pmatrix} \cosh \gamma_1 x & \sinh \gamma_2 x \\ \gamma_1 \sinh \gamma_1 x & \gamma_2 \cosh \gamma_2 x \end{pmatrix} .
\]

(173)

The potential may be reduced to the form

\[
V_{2SR} = -\left(\gamma_2^2 - \gamma_1^2\right) \frac{\gamma_2^2 \cosh^2 \gamma_1 x + \gamma_1^2 \sinh^2 \gamma_2 x}{(\gamma_2 \cosh \gamma_2 x \cosh \gamma_1 x - \gamma_1 \sinh \gamma_2 x \sinh \gamma_1 x)^2} .
\]

(174)

The normalized eigenfunctions of $V_{2SR}$ may be written in the form

\[
\tilde{\psi}_{2SR}(E_2) = \left(\frac{\gamma_2}{2} \left(\gamma_2^2 - \gamma_1^2\right)\right)^{1/2} \frac{\cosh \gamma_1 x}{\det D_{2SR}} ,
\]

\[
\tilde{\psi}_{2SR}(E_1) = \left(\frac{\gamma_1}{2} \left(\gamma_2^2 - \gamma_1^2\right)\right)^{1/2} \frac{\sinh \gamma_2 x}{\det D_{2SR}} .
\]

(175)

In terms of these normalized eigenfunctions the symmetric reflectionless potential may be written in the form [36]

\[
V_{2SR} = -2 \left[\gamma_2 \tilde{\psi}_{2SR}(E_2) + \gamma_1 \tilde{\psi}_{2SR}(E_1)\right] .
\]

(176)

Fig. 7 shows some examples of symmetric reflectionless potentials with two bound states for certain choices of bound state energies.

Certain features of $V_{2SR}$ may be analytically established:

(i) if $E_2 > 3E_1$ then $x = 0$ is a minimum of the potential and there are no additional minima and $V_{2SR}$ is a single well. In particular if $E_2 = 4E_1$ the resulting potential has the simple form

\[
V_{2SR} = -3\gamma_1^2 \text{sech}^2 \gamma_1 x ,
\]

(177)

i.e., $V_{2SR}$ is a sech$^2$ potential with bound states at $-2\gamma_1^2$ and $-\gamma_1^2/2$.

(ii) if $E_2 < 3E_1$ then $x = 0$ is a maximum of the potential and there is a pair of additional minima for $|x| \neq 0$ and $V_{2SR}$ under these conditions is a symmetric double well. These features are illustrated in Fig. 7.
FIGURE 7. Symmetric reflectionless potentials with bound states at energies $E_1 = -\gamma_1^2/2$ and $E_2 = -\gamma_2^2/2$ for $\gamma_1 = 1$ and $\gamma_2 = (a) 1.1, (b) 1.3, (c) 1.5, (d) 2.0$. The locations of the bound levels are indicated by broken lines.

4.3. Potentials with an arbitrary number of bound states

By an extension of the procedure outlined in the last two sections it is possible to construct a hierarchy of Hamiltonians with successively increasing number of bound states starting from the Hamiltonian $H_0$ with no bound states [21, 36]. Denoting the Hamiltonian with $n$ bound states by $H_n$ and the ground state energy of $H_n$ by $E_n$:

$$E_n = -\gamma_n^2/2, \quad \gamma_n^2 > \gamma_{n-1}^2 > \ldots > \gamma_1^2,$$

the Hamiltonian hierarchy is given by

$$H_m = A_{m-1}^{-}(E_m)A_{m-1}^{+}(E_m) + E_m$$

$$= H_{m-1} + [A_{m-1}^{-}(E_m), A_{m-1}^{+}(E_m)], \quad m = 1, 2, \ldots, n,$$

where

$$A_{m-1}^{\pm}(E_m) = \frac{1}{\sqrt{2}} \left[ \pm \frac{d}{dx} + \frac{d}{dx} \ln \psi_{m-1}(E_m) \right],$$

and $\psi_{m-1}(E_m)$ is a non-normalizable nodeless solution of the eigenvalue equation for $H_{m-1}$ at energy $E_m$ which lies below the ground state of $H_{m-1}$. The potentials in the
hierarchy are related by

\[ V_m = V_{m-1} - \frac{d^2}{dx^2} \ln \psi_{m-1}(E_m) . \]  \hspace{1cm} (181)

The ground state eigenfunction of \( H_m \) is given by

\[ \psi_m(E_m) \sim \frac{1}{\psi_{m-1}(E_m)} , \]  \hspace{1cm} (182)

while all the other eigenfunctions of \( H_m \) are given in terms of the eigenfunctions of \( H_{m-1} \) by

\[ \psi_m(E_i) \sim A_{m-1}^{-1}(E_m)\psi_{m-1}(E_i) , \quad i = 1, 2, \ldots, m-1 , \quad m = 1, 2, \ldots, n . \]  \hspace{1cm} (183)

This network of interrelated eigenfunctions can be disentangled to express all eigenfunctions of \( H_m \) in terms of the solutions in the reference potential \( V_0 \). Iteration of Eq. (181) shows that the potential with \( n \) bound states is related to \( V_0 \) by

\[ V_n = V_0 - \frac{d^2}{dx^2} \left[ \ln \psi_0(E_1)\psi_1(E_2)\ldots\psi_{n-1}(E_n) \right] . \]  \hspace{1cm} (184)

It is possible to express the product of eigenfunctions in the above equation in terms of the solutions \( \psi_0(E_i) \) in the potential \( V_0 \) for various energies \( E_i \). It is then possible to express \( V_n \) in the form

\[ V_n = V_0 - \frac{d^2}{dx^2} \ln \det D_n , \]  \hspace{1cm} (185)

where the matrix \( D_n \) is given by

\[ [D_n]_{jk} = \frac{d^{j-1}}{dx^{j-1}} \psi_0(E_k) , \quad j, k = 1, 2, \ldots, n . \]  \hspace{1cm} (186)

The eigenfunctions for the potential \( V_n \) may be expressed in the form

\[ \psi_n(E_i) \sim [D_n^{-1}]_{in} , \quad i = 1, 2, \ldots, n . \]  \hspace{1cm} (187)

The proof that the elements in the last column of the inverse of the matrix \( D_n \) are indeed the eigenfunctions for \( V_n \) in Eq. (185) with eigenenergies \( E_i, i = 1, 2, \ldots, n \), is straightforward but involves a long algebraic calculation. The requirement that \( \psi_{m-1}(E_m), m = 1, 2, \ldots, n \), be nodeless can be met by choosing the non-normalizable solutions \( \psi_0(E_m), m = 1, 2, \ldots, n \), such that the determinant of \( D_n \) has no zeros. The eigenfunction relation given in Eq. (183) can be extended to positive energies to show that the reflection coefficient of \( V_m \) is related to the reflection coefficient of \( V_{m-1} \) by

\[ R_m(k) = \frac{\gamma_m - ik}{\gamma_m + ik} R_{m-1}(k) . \]  \hspace{1cm} (188)
Iteration of this relation gives

\[ R_m(k) = \left[ \prod_{m=1,2,\ldots,n} \left( \frac{\gamma_m - ik}{\gamma_m + ik} \right) \right] R_0(k) . \quad (189) \]

Eqs. (185)-(187) provide a recipe for constructing potentials with bound states at specified energies \( E_m \) and reflection coefficient for positive energies given by Eq. (189).

The algorithm for constructing reflectionless potentials with \( n \) bound states is a particular case of the procedure given above corresponding to the choice \( V_0 = 0 \). Since \( R_0(k) = 0 \) when \( V_0 = 0 \), \( R_n(k) \) also vanishes. The free particle solutions at energies \( E_j \) are given by

\[ \psi_0(E_j) = \frac{1}{2} \left[ \exp \left( \gamma_j x + \theta_j \right) + (-)^{j+1} \exp \left( -\gamma_j x - \theta_j \right) \right] , \quad (190) \]

where \( \theta_j \) are arbitrary phase factors. For odd values of \( j \), \( \psi_0(E_j) \) is a \( \cosh \) function and hence nodeless while for even values of \( j \), \( \psi_0(E_j) \) is a \( \sinh \) function with a single node. Such a choice of solutions ensures that \( \det D_n \) has no zeros. A symmetric reflectionless potential with \( n \) bound states may be obtained by choosing \( \theta_j = 0 \) for all values of \( j \). The potential so obtained is given by

\[ V_{nSR} = -\frac{d^2}{dx^2} \ln \det D_{nSR} , \quad (191) \]

where the elements of the matrix \( D_{nSR} \) are given by

\[ [D_{nSR}]_{jk} = \frac{(-1)^{j-1}}{2} \left[ \exp(\gamma_j x) + (-1)^{j+k} \exp(-\gamma_k x) \right] . \quad (192) \]

The unnormalized eigenfunctions of this potential may be given in terms of the elements in the column \( n \) of the inverse of the matrix \( D_{nSR} \) and the normalized eigenfunctions may be written in the form

\[ \tilde{\psi}_{nSR}(E_i) = \left( \frac{\gamma_i}{2} \prod_{k \neq i} |\gamma_k^2 - \gamma_i^2| \right)^{1/2} \left[ D_{nSR}^{-1} \right]_{in} , \quad i = 1, 2, \ldots, n . \quad (193) \]

The relationship of this representation of the symmetric reflectionless potentials to other apparently different representations of the same potential can be established.

It can be shown [36] that the symmetric reflectionless potential with \( n \) bound states may be represented in terms of the normalized bound state eigenfunctions in the form

\[ V_{nSR} = -2 \sum_{j=1}^{n} \left[ \gamma_j \tilde{\psi}_{nSR}^2(E_j) \right] , \quad (194) \]

which is a generalized form of the results given in Eqs. (157) and (176) for the cases \( n = 1 \) and \( n = 2 \).
The analysis of Kay and Moses [37] and the \( n \)-soliton solution of the Korteweg-deVries equation [38, 39, 40] lead to the result that the symmetric reflectionless potential may be expressed in terms of a matrix \( M \) with elements

\[
M_{jk} = \delta_{jk} + \frac{\lambda_j(x)\lambda_k(x)}{\gamma_j + \gamma_k},
\]

(195)

where \( \delta_{jk} \) is the Kronecker delta function and

\[
\lambda_j(x) = C_j \exp(-\gamma_j x), \quad \frac{C_j^2}{2\gamma_j} = \prod_{k \neq j} \left( \frac{\gamma_k + \gamma_j}{\gamma_k - \gamma_j} \right),
\]

(196)

in the form

\[
V = -\frac{d^2}{dx^2} \ln \det M.
\]

(197)

It can be shown that the matrices \( M \) and \( D_{nSR} \) are related by

\[
M = 2GA^{-1}D_{nSR}G^{-1},
\]

(198)

where the elements of \( G \) and \( A \) are given by

\[
G_{ij} = \delta_{ij} \left( \frac{\gamma_j}{2} \prod_{k \neq j} |\gamma_k^2 - \gamma_j^2| \right)^{1/2},
\]

\[
A_{ij} = \gamma_j^{-1} \exp(\gamma_j x).
\]

(199)

It is clear that

\[
\det A \propto \exp \left( \sum_i \gamma_i x \right).
\]

(200)

Using Eqs. (198)-(200) it is easy to show that

\[
\frac{d^2}{dx^2} \ln \det M = \frac{d^2}{dx^2} \ln \det D_{nSR},
\]

(201)

leading to the result that the instantaneous \( n \)-soliton solution of the KdV equation given by Eqs. (195)-(197) is identical to the symmetric reflectionless potential constructed using the methods of Supersymmetric Quantum Mechanics. These ideas will be further explored in the next subsection.

### 4.4. Backlund transformation, KdV equation and supersymmetry

A potential of the form given in Eq. (154) can be regarded as the instantaneous one-soliton solution \( V_1(x) = v_1(x,0) \) of the Korteweg-deVries equation [38] in the form

\[
\left( \frac{\partial^3}{\partial x^3} - 12v \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) v(x,t) = 0,
\]

(202)
with \( v = v_1 \). The solution for all \( t \) is
\[
v_1(x,t) = -\gamma_1^2 \text{sech}^2 \left( \gamma_1 x - 4\gamma_1^2 t - \tanh^{-1} \alpha_1 \right) .
\] (203)
and the soliton travels with a velocity proportional to its amplitude. The \( n \)-soliton solution of the KdV equation (202) has been studied extensively [37, 39, 40]. The time dependent \( n \)-soliton solution may be given in the form of Eqs. (195)-(197) with the choice of
\[
\lambda_j(x) = C_j \exp \left( -\gamma_j x + 4\gamma_j^3 t \right) ,
\] (204)
by endowing the basis functions with a time dependence of a specific form. It is this specific time dependence that guarantees that the \( n \)-soliton solution satisfies the partial differential Eq. (202). Similarly, a time dependence for the same potential constructed using SUSY and given by Eqs. (184)-(186) and (190) can be introduced by allowing the phases \( \theta_j \) to have a time dependence of the form
\[
\theta_j(t) = \theta_j(0) - 4\gamma_j^3 t .
\] (205)
Such a choice of time dependence guarantees that the potential \( V_n \) satisfies the KdV equation (202). It is possible to verify these assertions by a long calculation.

It is possible to consider other choices of time dependence for the phases. It can be shown that if the phases are allowed to vary with time in the form
\[
\theta_j(t) = \theta_j(0) - 2^{m-1}\gamma_j^m t ,
\] (206)
where the index \( m \) can take all odd values \( \geq 3 \) then each of these cases can lead to a solution of a member of the Kadomtsev-Petviashvili (KP) hierarchy studied by Caudrey et al. [41]. \( m = 3 \) leads to the KdV equation (202) while \( m = 5,7,\ldots \) lead to higher members of the KP hierarchy (Sukumar [42]). For this class of non-linear equations the \( n \)-soliton solution is given by Eqs. (185), (186), (190) and (206). For example \( m = 5 \) leads to the non-linear equation
\[
\left( \frac{\partial^5}{\partial x^5} - 20V \frac{\partial^3}{\partial x^3} - 40 \frac{\partial V}{\partial x} \frac{\partial^2}{\partial x^2} + 120V^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) V(x,t) = 0 ,
\] (207)
which has been studied by Sawada and Kotera [43] and Caudrey et al [41].

In the context of the KdV equation the transformations of solutions possessing \( n - 1 \) solitons to those with \( n \) solitons are known as Backlund transformations [44]. The Backlund transformation for the KdV equation can be understood as follows. If we consider a function \( F(x,t) \) satisfying a modified KdV equation of the form
\[
G(x,t) = \frac{\partial F}{\partial t} + 12 \left( \gamma^2 - F^2 \right) \frac{\partial F}{\partial x} + \frac{\partial^3 F}{\partial x^3} = 0 ,
\] (208)
then it can be established that the function \( v(x,t) \) defined by
\[
v(x,t) = \frac{1}{2} \left( F^2 - \frac{\partial F}{\partial x} - \gamma^2 \right) ,
\] (209)
can be shown to satisfy the KdV equation since Eq. (202) is equivalent to the equation

\[ 2FG - \frac{\partial G}{\partial x} = 0, \]  

(210)

which is clearly satisfied because \( F \) satisfies Eq. (208). Also it is evident that by similar reasoning

\[ 2FG + \frac{\partial G}{\partial x} = 0, \]  

(211)

which implies that the function \( \tilde{v}(x,t) \) is defined by

\[ \tilde{v}(x,t) = \frac{1}{2} \left( F^2 + \frac{\partial F}{\partial x} - \gamma^2 \right), \]  

(212)

also satisfies the KdV equation (202). With suitable boundary conditions on \( F \), one can interpret \( \tilde{v} \) as an \( n \)-soliton solution if \( v \) is an \( (n-1) \)-soliton solution. By eliminating \( F \) the relation between \( v \) and \( \tilde{v} \) may be established to be

\[ \frac{\partial}{\partial x} \left[ \left( v + \tilde{v} + \gamma^2 \right)^{1/2} \right] = \tilde{v} - v. \]  

(213)

We now show that this relation between \( v \) and \( \tilde{v} \) is consistent with the relation between supersymmetric partner potentials defined by Eqs. (178)-(182). If we define the function

\[ F(x) = \frac{d}{dx} \ln \psi_m(E_m), \]  

(214)

then using Eqs. (178)-(182) it is easy to show that the potentials \( V_n \) and \( V_{n-1} \) can be written in terms of \( F \) in the form

\[ V_n = \frac{1}{2} \left( F^2 + \frac{d}{dx} F - \gamma_n^2 \right), \]  

\[ V_{n-1} = \frac{1}{2} \left( F^2 - \frac{d}{dx} F - \gamma_n^2 \right). \]  

(215)

The above equations are identical to Eqs. (209) and (212) which define \( v \) and \( \tilde{v} \). Thus Supersymmetric Quantum Mechanics enables the identification of the function \( F \) defined by the Backlund transformation (Eq. (209)) for the KdV equation as the logarithmic derivative of the ground state eigenfunction of the potential \( V_n(x,t) \) which supports \( n \) bound states where the time dependence arises from the time dependence of the \( \theta_j \) in Eqs. (190) and (205). The above argument can be extended to interpret the Backlund transformation for the entire system of non-linear equations defined by the KP hierarchy.

### 4.5. Summary

We have shown that by repeatedly using the algebra of supersymmetry in a step by step fashion it is possible to construct potentials with bound states at arbitrary
energies. It has been shown that the non-normalizable solutions in a reference potential which supports no bound states constitute the input in this construction. The reflection coefficient of the potential with \( n \) bound states constructed by this procedure is related to the reflection coefficient in the reference potential. \( V_n \) is in general not only a function of the \( n \) bound state energies \( E_i \) but also a function of \( n \) parameters \( \theta_i, i = 1, 2, \ldots, n \). \( \theta_i \) characterizes a particular linear superposition of the two linearly independent non-normalizable solutions in the reference potential \( V_0 \) at energy \( E_i \). \( \theta_i \) can take such values that ensure that the determinant of \( D_n \) is free of zeros.

By choosing the reference potential to be \( V_0 = 0 \) reflectionless potentials with \( n \) bound states may be constructed. The reflectionless potential so obtained is not necessarily a symmetric function of \( x \). By choosing the parameters \( \theta_i \) to have specific values symmetric reflectionless potentials can be constructed. The resulting symmetric potential is identical to the one constructed from the \( n \)-soliton solution of the KdV equation. Symmetric reflectionless potentials are unique and interesting because they are specified entirely by their bound states. They have been shown to provide good approximations to confining potentials, such as those which confine quarks, in the range of energies actually probed by the levels (Thacker et al. [45], Quigg and Rosner [46], Kwong and Rosner [47]). Kwong and Rosner have constructed an approximation to a quarkonium potential on the basis of information about its \( nS \) levels. Using the energies and leptonic widths of 10 levels below the flavour threshold Kwong and Rosner have used the algorithm for constructing symmetric reflectionless potentials to construct a potential \( V(r) \). Calculations such as these provide information on the distance scale over which \( t\bar{t} \) bound state information is likely to shed light on the interquark force.

We have shown that it is possible to identify a hierarchy of non-linear equations of the KdV type which have the property that the \( n \)-soliton solution of these equations can be explicitly given in analytic form. The connection between SUSYQM and Backlund transformations provides interesting insights and enables a physical interpretation of the function that is used to construct the Backlund transformation. The symmetries which lead to an infinite number of conservation laws in integrable systems such as the KdV equation have already been recognized as being related to Kac-Moody algebras. It is clear that we can now add supersymmetry to the list of properties connected with such systems.

We have also shown that the symmetric reflectionless potentials may be expressed in terms of the normalized bound state eigenfunctions in a particularly simple form. It is clear that the construction based on supersymmetry not only agrees with calculations based on other procedures but also provides unique insight into the structure of symmetric reflectionless potentials and the structure of the multi-soliton solutions of the KdV hierarchy and the Backlund transformations associated with these non-linear equations.

5. PAIRS OF SUSY TRANSFORMATIONS FOR THE RADIAL SCHRÖDINGER EQUATION

It was shown in an earlier lecture that it is possible to identify four different transformations by which one can find a supersymmetric partner to a given radial Schrödinger equa-
tion. The modifications of the Jost function and the singularity structure in the asymptotic region $r \to \infty$ and in the region $r \to 0$ for the 4 types of SUSY transformations were identified. Each of the transformations alters the spectral density in a specific manner and either removes or adds a bound state or maintains the same spectrum. In this lecture we study how pairs of SUSY transformations may be used to modify or maintain the spectrum with or without altering the spectral density. An exactly solvable example is used to illustrate the procedure.

5.1. Classification of the four types of SUSY transformations

In §2 four different transformations of the radial Schrödinger equation were identified by $T_1$, $T_2$, $T_3$ and $T_4$. The potentials, eigenstates and phase shifts and the Jost function after the supersymmetric transformation were denoted by adding a tilde. The different types of transformations were distinguished by adding a suffix. In this lecture we adopt the same notations for single transformations. Successive transformations will be indicated by adding further suffixes and tildes. We also use the same notation as in §2 that the regular solution at a given energy $\tilde{E} = -\tilde{\gamma}^2/2$ will be denoted by $\phi(r)$ and the Jost solution by $\eta(r)$. It was shown that these functions obey the boundary conditions

$$\lim_{r \to 0} \phi(r) = \frac{r^{(l+1)}}{(2l+1)!!}, \quad \lim_{r \to \infty} \phi(r) \sim \exp(\tilde{\gamma} r),$$

$$\lim_{r \to 0} \eta(r) \sim r^{-l}, \quad \lim_{r \to \infty} \eta(r) \sim \exp(-\tilde{\gamma} r).$$

We also use the notation $\psi^{(m)}(r)$ is an abbreviation for $\psi(r,E^{(m)})$.

It was shown in §2 that for all the four types of transformations the potential is unaltered in the region $r \to \infty$ and that each of the transformations produces an alteration of the potential in the region of small and medium $r$ and near $r \to 0$ the alteration of the potential is equivalent to altering the angular momentum $l$ in the centrifugal part $l(l+1)/r^2$ of the potential $V(r)$ corresponding to a definite partial wave. The changes for the four transformations corresponding to the factorization energy $E = -\gamma^2/2$ were identified as:

$$T_1: \quad \tilde{F} = F \frac{k}{k - i\gamma}, \quad \lim_{r \to 0} l \to l + 1, \quad \tilde{\delta} = \delta - \tan^{-1}(\gamma/k) - \frac{\pi}{2},$$

$$T_2: \quad \tilde{F} = F \frac{k - i\gamma}{k}, \quad \lim_{r \to 0} l \to l - 1, \quad \tilde{\delta} = \delta + \tan^{-1}(\gamma/k) + \frac{\pi}{2},$$

$$T_3: \quad \tilde{F} = F \frac{k}{k + i\gamma}, \quad \lim_{r \to 0} l \to l + 1, \quad \tilde{\delta} = \delta + \tan^{-1}(\gamma/k) - \frac{\pi}{2},$$

$$T_4: \quad \tilde{F} = F \frac{k + i\gamma}{k}, \quad \lim_{r \to 0} l \to l - 1, \quad \tilde{\delta} = \delta - \tan^{-1}(\gamma/k) + \frac{\pi}{2}. \quad (218)$$

$T_1$ eliminates the ground state at $E$ leaving the rest of the spectrum of eigenvalues unaltered. $T_2$ adds a new ground state at $E$ while leaving the rest of the spectrum unaltered. $T_3$ and $T_4$ maintain the same spectrum and alter the Jost function and the
singularity of the potential at the origin. These four transformations may be viewed as the building blocks which can be used in suitable combinations to produce desired modifications of the spectrum, the Jost function and the singularity at the origin. We next study pairs of SUSY transformations and the alteration of the spectrum and the Jost function that they produce.

5.2. Pairs of SUSY transformations

We consider two successive transformations performed at the same energy $E$. Of the sixteen possibilities that arise from a combination of any one of the $T_j$ with any other $T_k$, it is clear when $j$ and $k$ are different the combination $T_j T_k$ is equivalent to $T_k T_j$. Of the remaining ten possibilities $T_1 T_1$ does not exist because it is not possible to remove the same bound state twice and $T_2 T_2$ does not exist because the same bound state can not be added twice. Therefore there are eight distinct possibilities for pairs of SUSY transformations which must be considered. We now classify them.

(1) $T_1$ followed by $T_2$ removes a bound state at energy $E$ and adds the same bound state at the same energy. At the end of the two transformations we get a potential which has the same spectrum as the original potential but the normalization constant of the ground state may be altered and the Jost function is unaltered.

$$\tilde{F}_{12} = F.$$  \hspace{1cm} (219)

The phase shifts and the normalization constants of all the excited state eigenfunctions are unaltered. Therefore $T_2 T_1$ produces the phase-equivalent family corresponding to a given potential.

(2) $T_1$ followed by $T_3$ removes a bound state at energy $E$ and alters the spectral density and the normalization constants of the remaining eigenstates. The Jost function is modified to

$$\tilde{F}_{13} = F \frac{k^2}{k^2 + \gamma^2}.$$ \hspace{1cm} (220)

The phase shifts for positive energies are unaltered. However, the resulting potential is singular with a singularity of the form $r^{-2}$.

(3) $T_1$ followed by $T_4$ removes a bound state at energy $E$ and keeps the spectral density unaltered which implies that the normalization constants of all the other bound states are unaltered. The Jost function is modified to

$$\tilde{F}_{14} = F \frac{k + i\gamma}{k - i\gamma}.$$ \hspace{1cm} (221)

The phase shifts for positive energies will be altered.

(4) $T_3$ followed by $T_2$ adds a bound state below the ground state of the original potential without altering the spectral density. The normalization constants of all the other states are left unaltered. The Jost function is modified to

$$\tilde{F}_{32} = F \frac{k - i\gamma}{k + i\gamma}.$$ \hspace{1cm} (222)
The phase shifts for positive energies will, however, be altered.

(5) $T_2$ followed by $T_4$ adds a bound state below the ground state of the original potential and alters the spectral density and the normalization constants of the other eigenstates. The Jost function is modified to

$$\tilde{F}_{24} = F \frac{k^2 + \gamma^2}{k^2}.$$ (223)

The phase shifts for positive energies will not be altered. Hence $T_4T_2$ produces a family of phase-equivalent potential with a new ground state and different members of the family will have different values for the normalization constants of the ground state. The resulting potential is singular at the origin.

(6) $T_3$ followed by another $T_3$ maintains the same spectrum and alters the spectral density. The normalization constants of all the states are altered and the resulting potential has $r^{-2}$ singularity at the origin. The Jost function is modified to

$$\tilde{F}_{33} = F \left( \frac{k}{k + i\gamma} \right)^2.$$ (224)

The phase shifts for positive energies and the normalization constants of the eigenfunctions will be altered.

(7) $T_3$ followed by $T_4$ transforms the original Hamiltonian back to itself without altering the spectral density, spectrum and the phase shifts for positive energies. In fact

$$\tilde{F}_{34} = F.$$ (225)

(8) $T_4$ followed by another $T_4$ is similar to case (6) and also maintains the same spectrum, alters the spectral density and the normalization constants of all the states. The resulting potential is a singular potential. The Jost function is modified to

$$\tilde{F}_{44} = F \left( \frac{k + i\gamma}{k} \right)^2.$$ (226)

The phase shifts for positive energies and the normalization constants of the eigenfunctions will be altered.

The eight cases listed above exhaust the possible combinations of two SUSY transformations. We now examine these cases in detail.

(1) We now show that $T_1$ followed by $T_2$ generates the phase-equivalent family to a given potential

$$V(r) = \frac{l(l+1)}{2r^2} + v(r),$$ (227)

with ground state $\psi^{(0)}(r)$ at energy $E^{(0)}$. After the first transformation $T_1$ which eliminates the ground state of $V$ the potential is

$$\tilde{V}_1(r) = V(r) - \frac{d^2}{dr^2} \ln \psi^{(0)}(r).$$ (228)
Using the second transformation $T_2$ the eliminated state can be introduced as the new ground state below the ground state of $\tilde{V}_1$. It was shown in §2 that the solution in $\tilde{V}_1$ at the corresponding energy may be given in the form

$$\tilde{\psi}_1(r, E(0), \alpha) = \tilde{\phi}_1(r, E(0)) \cos \alpha + \tilde{\eta}_1(r, E(0)) \sin \alpha,$$

where $\tilde{\phi}_1$ is the regular solution in $\tilde{V}_1$ at energy $E(0)$ and $\tilde{\eta}_1$ is the Jost solution in $\tilde{V}_1$ at the same energy and are given by

$$\tilde{\phi}_1 \sim \frac{1}{\psi^{(0)}(r)} \int_0^r (\psi^{(0)}(x))^2 \, dx,$$

$$\tilde{\eta}_1 \sim \frac{1}{\psi^{(0)}(r)} \int_r^\infty (\psi^{(0)}(x))^2 \, dx. \quad (230)$$

$\tilde{\psi}$ can be written in terms of the parameter $\lambda$ defined by

$$\tan \alpha = \frac{1}{\lambda + 1}, \quad (231)$$

in the form

$$\tilde{\psi}_1(r, E(0), \lambda) = \frac{1 + \lambda \int_0^r (\psi^{(0)}(x))^2 \, dx}{\psi^{(0)}(r)}, \quad -1 < \lambda < \infty. \quad (232)$$

The potential at the end of the two transformations is given by

$$\tilde{V}_{1,2}(r) = V(r) - \frac{d^2}{dr^2} \ln \left( 1 + \lambda \int_0^r (\psi^{(0)}(x))^2 \, dx \right), \quad (233)$$

which has a spectrum identical with that of $V(r)$ and has normalized ground state obtained from the inverse of $\tilde{\psi}$ in Eq. (232) in the form

$$\tilde{\psi}_{1,2}^{(0)} = \frac{(1 + \lambda)^{1/2} \psi^{(0)}(r)}{1 + \lambda \int_0^r (\psi^{(0)}(x))^2 \, dx}. \quad (234)$$

The excited states at the end of the two transformations are given by

$$\tilde{\psi}_{1,2}^{(m)} = - \left( E^{(m)} - E^{(0)} \right)^{-1} \tilde{A}_2^{-} \tilde{A}_1^{-} \psi^{(m)}, \quad m = 1, 2, \ldots, \quad (235)$$

where

$$\tilde{A}_2^{-} (E^{(0)}, \lambda) = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \tilde{\psi}_1 \right],$$

$$\tilde{A}_1^{-} = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \psi^{(0)}(r) \right]. \quad (236)$$
The excited state eigenfunctions can be simplified to the form
\[
\tilde{\psi}_{1,2}^{(m)} = \psi^{(m)} - \lambda \frac{\zeta^{(m)}(r)}{2(E^{(m)} - E^{(0)})},
\]
\[
\zeta^{(m)}(r) = \frac{\psi^{(0)}(r)}{1 + \lambda \int_0^r \left(\psi^{(0)}(x)\right)^2 dx} \left(\psi^{(m)} \frac{d}{dr} \psi^{(0)} - \psi^{(0)} \frac{d}{dr} \psi^{(m)}\right).
\] (237)

The Wronskian relation arising from the Schrödinger equation considered at two different energies in the form
\[
\left(\psi^{(m)} \frac{d}{dr} \psi^{(0)} - \psi^{(0)} \frac{d}{dr} \psi^{(m)}\right) = 2 \left(E^{(m)} - E^{(0)}\right) \int_0^r \psi^{(0)}(x) \psi^{(m)}(x) dx,
\] (238)
may then be used to express the excited state eigenfunctions in the form
\[
\tilde{\psi}_{1,2}^{(m)}(r) = \psi^{(m)}(r) - \lambda \psi^{(0)}(r) \frac{\int_0^r \psi^{(0)}(x) \psi^{(m)}(x) dx}{1 + \lambda \int_0^r \left(\psi^{(0)}(x)\right)^2 dx}, \quad m = 1, 2, \ldots.
\] (239)

From Eqs. (234) and (239) it can be established that
\[
\tilde{\delta}_{1,2}(l, k) = \delta(l, k),
\]
\[
\lim_{r \to 0} \tilde{\psi}_{1,2}^{(m)}(r, \lambda) = \lim_{r \to 0} \psi^{(m)}(r), \quad m = 1, 2, \ldots,
\]
\[
\lim_{r \to 0} \tilde{\psi}_{1,2}^{(0)}(r, \lambda) = (1 + \lambda)^{1/2} \lim_{r \to 0} \psi^{(0)}(r).
\] (240)

It is also clear from Eq. (218) that at the end of the two transformations $T_1$ and $T_2$ the singularity at the origin arising from the centrifugal part of the potential is unaltered and also that the phase shifts are unaltered. These results show that the family of potentials $\tilde{V}_{1,2}(r, \lambda)$ in Eq. (233) for $\infty > \lambda > -1$ have identical spectra, identical phase shifts and identical normalization constants for the excited states but have different normalization constants for the ground state for different values of $\lambda$. Hence this family of potentials belongs to a phase-equivalent family. These expressions for the new potential and the new eigenfunctions are in agreement with the results obtained using the Gelfand-Levitan procedure for changing the normalization constant of the ground state [26, 30]. Thus we have shown that the Gelfand-Levitan [24] procedure for changing the normalization constant of the ground state without changing the spectrum is equivalent to a transformation of the type $T_1$ followed by another suitable transformation of the type $T_2$ [28].

(2) We now consider $T_1$ followed by $T_3$. The first step is the same as case (1) and leads to the potential given by Eq. (228). The second step corresponds to a $T_3$ transformation with the choice of value $\alpha = 0$ in Eq. (229) which also corresponds to the choice $\lambda = \infty$ in Eq. (232). Such a choice for $\tilde{\psi}_1$ leads to the new potential after the two transformations of the form
\[
\tilde{V}_{1,3}(r) = V(r) - \frac{1}{2} \ln \left[\int_0^r \left(\psi^{(0)}(x)\right)^2 dx\right],
\] (241)
which has a spectrum identical with that of $V(r)$ except for missing the ground state of $V$. The eigenfunctions of $\tilde{V}_{1,3}$ are related to the excited state eigenfunctions of $V$ by

$$\tilde{\psi}_{1,3}^{(m)} = -\left(E^{(m)} - E^{(0)}\right)^{-1} \tilde{A}_3 A_1^{-1} \psi^{(m)}, \quad m = 1, 2, \ldots ,$$  \hspace{1cm} (242)

where

$$\tilde{A}_3 = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \left( \frac{1}{\psi^{(0)}(r)} \int_0^r \left( \psi^{(0)}(x) \right)^2 dx \right) \right],$$  \hspace{1cm} (243)

and $A_1^{-1}$ has the same form as in Eq. (236). From these equations it can be established using the same steps as in case (1) that

$$\tilde{\psi}_{1,3}^{(m)} = \psi^{(m)}(r) - \psi^{(0)}(r) \frac{\int_0^r \psi^{(0)}(x) \psi^{(m)}(x) dx}{\int_0^r \left( \psi^{(0)}(x) \right)^2 dx}, \quad m = 1, 2, \ldots ,$$  \hspace{1cm} (244)

and that the normalization constants of the eigenfunctions are altered. It can be shown from Eq. (241) or from Eq. (218) that

$$\lim_{r \to 0} \tilde{V}_{1,3}(r) = \frac{(l+2)(l+3)}{2r^2}, \quad \lim_{r \to \infty} \tilde{V}_{1,3}(r) = \frac{l(l+1)}{2r^2}$$  \hspace{1cm} (245)

which shows that the new potential is singular at the origin and has a short range repulsive singularity. Eq. (218) can also be used to show that the phase shifts are decreased by a constant amount $\pi$ for all positive energies which is equivalent to saying that the phase shifts are unaltered within modulo $\pi$. Thus a transformation of the type $T_1$ followed by a transformation of the type $T_3$ can be used to produce a singular potential which has the same phase shifts as the original potential but has one less bound state [48].

(3) We next consider $T_1$ followed by $T_4$. The first step is the same as in cases (1) and (2) and leads to the potential given in Eq. (228). The second step corresponds to a $T_4$ transformation with the choice of value $\alpha = \pi/2$ in Eq. (229) which also corresponds to the choice $\lambda = -1$ in Eq. (232). Such a choice of $\tilde{\psi}_1$ leads to a new potential after the two transformations of the form

$$\tilde{V}_{1,4}(r) = V(r) - \frac{d^2}{dr^2} \ln \left( \int_0^\infty \left( \psi^{(0)}(x) \right)^2 dx \right),$$  \hspace{1cm} (246)

which has the same spectrum as $V$ except for missing the ground state of $V$. The eigenstates of the new potential can be obtained by the same procedure as for the earlier cases and can be readily obtained by taking $\lambda = -1$ in Eq. (239) to give

$$\tilde{\psi}_{1,4}^{(m)}(r) = \psi^{(m)}(r) - \psi^{(0)}(r) \frac{\int_r^\infty \psi^{(0)}(x) \psi^{(m)}(x) dx}{\int_r^\infty \left( \psi^{(0)}(x) \right)^2 dx}, \quad m = 1, 2, \ldots .$$  \hspace{1cm} (247)

It can be established from this equation that the bound state normalizations are unaffected. It can be shown using Eq. (218) that the phase shift relation for positive energies
\[
\delta_{1,4}(l, k) = \delta(l, k) - 2 \tan^{-1}\left(\frac{\gamma(0)}{k}\right).
\]  

(248)

It can also be shown using Eq. (218) that the singularity of the potential \( \tilde{V}_{1,4} \) at the origin is unaltered. The expressions for the new potential and the new eigenfunctions [28] are identical to the results given by Abraham and Moses [26]. Thus the Gelfand-Levitan procedure to eliminate the ground state without introducing additional singularities at the origin is equivalent to a transformation \( T_1 \) which eliminates the ground state followed by a transformation \( T_4 \) which removes the additional singularity at the origin which is introduced by the first transformation. The resultant potential after the two transformations is non-singular.

(4) We next consider \( T_3 \) followed by \( T_2 \). The transformation \( T_3 \) may be used at energy \( \tilde{E} \) below the ground state of \( V \) to produce a new potential which has the same spectrum as that of \( V \) as described in §2.6. The \( T_3 \) transformation is implemented by considering the regular solution \( \phi(r, \tilde{E}) \) for the potential \( V(r) \). After the \( T_3 \) transformation the new potential is

\[
\tilde{V}_3(r, \tilde{E}) = V(r) - \frac{d^2}{dr^2} \ln \phi(r, \tilde{E}).
\]  

(249)

We now apply a \( T_2 \) transformation at the energy \( \tilde{E} \) to introduce a new bound state below the ground state of \( V \). One of the solutions at energy \( \tilde{E} \) in the potential \( \tilde{V}_3 \) is given by

\[
\xi = \frac{1}{\phi(r, \tilde{E})},
\]  

(250)

and the second linearly independent solution is, therefore, given by

\[
\chi(r, \tilde{E}) = \frac{1}{\phi(r, \tilde{E})} \int_0^r \left( \phi(x, \tilde{E}) \right)^2 dx.
\]  

(251)

By studying the limiting behaviour of \( \phi, \xi \) and \( \chi \) it can be established that the regular and the Jost solution in \( \tilde{V}_3 \) are indeed proportional to \( \chi \) and \( \xi \) respectively. Hence using the general solution

\[
\tilde{\psi}_3(r, \tilde{E}, \alpha) = \chi(r, \tilde{E}) \sin \alpha + \xi(r, \tilde{E}) \cos \alpha,
\]  

(252)

the new potential after the two transformations may be given in the form

\[
\tilde{V}_{3,2}(r, \tilde{E}, \alpha) = V(r) - \frac{d^2}{dr^2} \ln \left( \cos \alpha + \sin \alpha \int_0^r \left( \phi(x, \tilde{E}) \right)^2 dx \right),
\]  

(253)

which has a ground state eigenvalue at \( \tilde{E} \) while all the other eigenvalues are identical to those of \( V \). The normalized ground state of \( \tilde{V}_{3,2} \) is given by

\[
\tilde{\psi}_{3,2}^{(0)}(r, \tilde{E}, \alpha) = \frac{(\sin \alpha \cos \alpha)^{1/2} \phi(r, \tilde{E})}{\cos \alpha + \sin \alpha \int_0^r \left( \phi(x, \tilde{E}) \right)^2 dx}.
\]  

(254)
Eq. (254) shows that the parameter $\alpha$ determines the normalization constant of the ground state. $\alpha = \frac{\pi}{4}$ corresponds to choosing the normalization constant as 1 and for this choice the resulting potential is

$$\tilde{V}_{3,2}(r, \tilde{E}, \frac{\pi}{4}) = V(r) - \frac{d^2}{dr^2} \ln \left( 1 + \int_0^r (\phi(x, \tilde{E}))^2 dx \right), \quad (255)$$

which has the ground state eigenfunction

$$\tilde{\psi}_{3,2}^{(0)}(r, \tilde{E}) = \frac{\phi(r, \tilde{E})}{1 + \int_0^r (\phi(x, \tilde{E}))^2 dx} . \quad (256)$$

The excited state eigenfunctions are given by

$$\tilde{\psi}_{3,2}^{(m)}(r, \tilde{E}) = - \left( E^{(m)} - E^{(0)} \right)^{-1} \tilde{A}^{-} A_{3}^{+} \psi^{(m)}(r), \quad m = 1, 2, \ldots , \quad (257)$$

where

$$\tilde{A}^{-} = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \left( \frac{1 + \int_0^r (\phi(x, \tilde{E}))^2 dx}{\phi(r, \tilde{E})} \right) \right],$$

$$A_{3}^{+} = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \phi(r, \tilde{E}) \right] . \quad (258)$$

Using the same method as in previous cases the expression for the excited state eigenfunctions can be brought to the form

$$\tilde{\psi}_{3,2}^{(m)}(r, \tilde{E}) = \psi^{(m)}(r) - \phi(r, \tilde{E}) \int_0^r \phi(x, \tilde{E}) \psi^{(m)}(x) dx \frac{1}{1 + \int_0^r (\phi(x, \tilde{E}))^2 dx}, \quad m = 1, 2, \ldots . \quad (259)$$

It is easy to show that

$$\lim_{r \to 0} \tilde{\psi}_{3,2}^{(m)}(r) = \lim_{r \to 0} \psi^{(m)}(r) . \quad (260)$$

Thus the normalization constants of the eigenfunctions are left unaltered. Eq. (218) can be used to show that the singularity of the potential at the origin is unaltered after the two transformations. It can also be shown that the phase shift relation for positive energies is

$$\tilde{\delta}_{3,2}(l, k) = \delta(l, k) + 2 \tan^{-1} \left( \frac{\tilde{\gamma}}{k} \right) . \quad (261)$$

The expressions for the new potential and the new eigenfunctions [28] derived above are in agreement with the results derived using the Gelfand-Levitan method [26].

The family of potentials $\tilde{V}_{3,2}(r, \tilde{E}, \alpha)$ in Eq. (253) lead to identical phase shifts and the normalization constants for the excited states of the potentials $\tilde{V}_{3,2}(r, \tilde{E}, \alpha)$ for various values of $\alpha$ in the range $0 < \alpha < \frac{\pi}{2}$, are the same. The parameter $\alpha$ affects
the normalization constant of the ground state. The family of potentials $\tilde{V}_{3,2}(r, \tilde{E}, \alpha)$ have identical spectra, identical phase shifts and identical normalization constants for all states except the ground state and is, therefore, a phase-equivalent family. Thus a transformation of the type $T_3$ followed by a transformation of the type $T_2$ enables the generation of a family of potentials which have a new ground state but without altering the spectral density and without altering the singularity of the potential at the origin.

(5) We next consider $T_4$ followed by $T_2$. This sequence is very similar to case (4) except that in the first step involving the transformation $T_4$ the Jost solution $\eta(r, \tilde{E})$ is used instead of the regular solution $\phi(r, \tilde{E})$ in the potential $V$. After the $T_4$ transformation the new potential is

$$\tilde{V}_4(r, \tilde{E}) = V(r) - \frac{d^2}{dr^2} \ln \eta(r, \tilde{E}).$$

(262)

We now apply a $T_2$ transformation at the energy $\tilde{E}$ to introduce a new bound state below the ground state of $V$. By identifying the general solution in $\tilde{V}_4$ at energy $\tilde{E}$ in the form

$$\tilde{\psi}_4(r, \tilde{E}, \beta) = \cos \beta + \sin \beta \int_r^\infty (\eta(x, \tilde{E}))^2 dx,$$

(263)

the new potential after the two transformations may be given in the form

$$\tilde{V}_{4,2}(r, \tilde{E}, \beta) = V(r) - \frac{d^2}{dr^2} \ln \left( \cos \beta + \sin \beta \int_r^\infty (\eta(x, \tilde{E}))^2 dx \right),$$

(264)

which has a ground state eigenvalue at $\tilde{E}$ while all the other eigenvalues are identical to those of $V$. The normalized ground state of $\tilde{V}_{4,2}$ is given by

$$\tilde{\psi}_{4,2}^{(0)}(r, \tilde{E}, \beta) = \frac{(\sin \beta \cos \beta)^{1/2} \eta(r, \tilde{E})}{\cos \beta + \sin \beta \int_r^\infty (\eta(x, \tilde{E}))^2 dx}.$$  

(265)

Eq. (265) shows that the parameter $\beta$ determines the normalization constant of the ground state. For $\beta = \pi/4$ the resultant potential after the two transformations is

$$\tilde{V}_{4,2}(r, \tilde{E}) = V(r) - \frac{d^2}{dr^2} \ln \left( 1 + \int_r^\infty (\eta(x, \tilde{E}))^2 dx \right),$$

(266)

which has the ground state eigenfunction

$$\tilde{\psi}_{4,2}^{(0)}(r, \tilde{E}) = \frac{\eta(r, \tilde{E})}{1 + \int_r^\infty (\eta(x, \tilde{E}))^2 dx}.$$  

(267)

The excited state eigenfunctions are given by

$$\tilde{\psi}_{4,2}^{(m)}(r) = - \left( E^{(m)} - E^{(0)} \right)^{-1} \tilde{A}_2 \tilde{A}_4 \psi^{(m)}(r), \quad m = 1, 2, \ldots,$$

(268)
where

\[
\tilde{A}_2^- = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \left( \frac{1 + \int_r^\infty (\eta(x, \tilde{E}))^2 dx}{\eta(r, \tilde{E})} \right) \right],
\]

\[
\tilde{A}_4^- = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dr} + \frac{d}{dr} \ln \eta(r, \tilde{E}) \right].
\]

(269)

Using the same methods as in the previous cases the excited state eigenfunctions can be shown to be given by

\[
\tilde{\psi}_{4,2}^{(m)}(r, \tilde{E}) = \psi_{4,2}^{(m)}(r) - \eta(r, \tilde{E}) \int_r^\infty \eta(x, \tilde{E}) \psi_{4,2}^{(m)}(x) dx \frac{1}{1 + \int_r^\infty (\eta(x, \tilde{E}))^2 dx}, \quad m = 1, 2, \ldots.
\]

(270)

It can be shown that the normalization constants of the eigenfunctions are left unaltered. It can also be shown using Eq. (218) that the phase shifts are increased by a constant amount \(\pi\) for all positive energies which is equivalent to saying that the phase shifts are unaltered within modulo \(\pi\). However

\[
\lim_{r \to 0} \tilde{\psi}_{4,2}(r) = \frac{(l - 2)(l - 1)}{2r^2}, \quad \lim_{r \to \infty} \tilde{\psi}_{4,2}(r) = \frac{l(l + 1)}{2r^2},
\]

(271)

which shows that \(\tilde{\psi}_{4,2}\) has a component which has a short range attractive \(r^{-2}\) singularity in addition to the usual centrifugal potential.

Thus we have shown that the application of a transformation of the type \(T_4\) followed by a transformation of the type \(T_2\) enables the construction of singular potentials which have the same phase shifts as the original potential but which have an additional bound state at a chosen energy. The family of potentials \(\tilde{V}_{4,2}(r, \tilde{E}, \beta)\), for various values of \(\beta\) in the range \(0 < \beta < \pi/2\), have identical phase shifts and identical normalization constants for the excited states but have different normalization constants for the ground state and belong to a family of phase-equivalent potentials.

(6) We now consider two successive \(T_3\) transformations performed at the same energy \(\tilde{E}\) below the ground state of \(V\). After the first \(T_3\) transformation using the regular solution \(\phi\) in the potential \(V\) the new potential \(\tilde{V}_3\) has the form given by Eq. (249). The second \(T_3\) transformation is performed using the choice \(\alpha = \pi/2\) in Eq. (252) so that \(\tilde{\psi}_3(r) = \chi(r)\) which is defined by Eq. (251). The resulting potential after the two transformations is then given by

\[
\tilde{V}_{3,3}(r, \tilde{E}) = V(r) - \frac{d^2}{dr^2} \ln \left[ \int_0^r (\phi(x, \tilde{E}))^2 dx \right],
\]

(272)

which leads to a spectrum of eigenvalues which is identical with that for \(V\). The transformed eigenstates may be shown to be given by

\[
\tilde{\psi}_{3,3}^{(m)}(r) = \psi_{3,3}^{(m)}(r) - \phi(r, \tilde{E}) \frac{\int_0^r \phi(x, \tilde{E}) \psi_{3,3}^{(m)}(x) dx}{\int_0^r (\phi(x, \tilde{E}))^2 dx}, \quad m = 0, 1, \ldots.
\]

(273)
It can be established that the normalization constants of the eigenfunctions and the spectral density are altered. Using Eq. (218) the phase shifts for positive energies can be shown to be related by
\[
\tilde{\delta}_{3,3}(l, k) = \delta(l, k) + 2\tan^{-1}\left(\frac{\tilde{\gamma}}{k}\right) - \pi .
\] (274)

It can also be shown that
\[
\lim_{r \to 0} \tilde{V}_{3,3}(r, \tilde{E}) = \frac{(l + 2)(l + 3)}{2r^2}, \quad \lim_{r \to \infty} \tilde{V}_{3,3}(r, \tilde{E}) = \frac{l(l + 1)}{2r^2},
\] (275)

which shows that \( \tilde{V} \) is a singular potential. Thus two successive \( T_3 \) transformations may be used to produce a new singular potential which has the same spectrum as \( V \) but has different phase shifts and different normalization constants.

(7) We next consider \( T_3 \) followed by \( T_4 \). The first step is the same as in case (6). In the second step the \( T_4 \) transformation is performed using the choice \( \alpha = 0 \) in Eq. (252) so that \( \tilde{\psi}_3(r) = \tilde{\xi}(r) \) defined by Eq. (250). With this choice it is clear that the potential after the two transformations is identical to the original potential \( V(r) \) since \( \phi(r)\tilde{\xi}(r) = 1 \). Thus \( T_4T_3 \) restores the original Hamiltonian and all the eigenstates and phase shifts are unaltered. Thus \( T_3 \) followed by \( T_4 \) performed at the same energy is equivalent to the identity transformation.

(8) The last combination we consider is two successive \( T_4 \) transformations at the same energy \( \tilde{E} \) below the ground state of \( V \). After the first \( T_4 \) transformation using the Jost solution \( \eta \) in the potential \( V \) the new potential \( \tilde{V}_4 \) has the form given by Eq. (262). The second \( T_4 \) transformation is performed by using the choice \( \beta = \pi/2 \) in Eq. (263) so that the new potential after the two transformations is given by
\[
\tilde{V}_{4,4}(r, \tilde{E}) = V(r) - \frac{d^2}{dr^2} \ln \left( \int_r^\infty (\eta(x, \tilde{E}))^2 \, dx \right),
\] (276)

which has the same spectrum as that for \( V(r) \). The transformed eigenfunctions may be shown to be given by
\[
\tilde{\psi}^{(m)}_{4,4}(r) = \psi^{(m)}(r) - \eta(r, \tilde{E}) \frac{\int_r^\infty \eta(x, \tilde{E}) \psi^{(m)}(x)}{\int_r^\infty (\eta(x, \tilde{E}))^2 \, dx}, \quad m = 0, 1, \ldots .
\] (277)

It can be established that the spectral density is altered and that the normalization constants of the eigenfunctions are altered. Using Eq. (218) it can be seen that the phase shifts for positive energies are related by
\[
\tilde{\delta}_{4,4}(l, k) = \delta(l, k) - 2\tan^{-1}\left(\frac{\tilde{\gamma}}{k}\right) + \pi .
\] (278)

It can also be shown that
\[
\lim_{r \to 0} \tilde{V}_{4,4}(r, \tilde{E}) = \frac{(l - 2)(l - 1)}{2r^2}, \quad \lim_{r \to \infty} \tilde{V}_{4,4}(r, \tilde{E}) = \frac{l(l + 1)}{2r^2},
\] (279)
which shows that $\tilde{V}_{4,4}$ is a singular potential. Thus two successive $T_4$ transformations may be used to produce a new singular potential which has the same spectrum as $V$ but has different phase shifts and different normalization constants.

The new eigenfunctions after two transformations given by Eqs. (239), (244), (247), (259), (270), (273) and (277) can all be expressed as the ratio of 2 determinants. The new potential can be expressed as a determinant of a matrix $M$. These identifications enable the extension of the procedure outlined in this lecture to the elimination or addition of an arbitrary number of states or the generation of isospectral Hamiltonians after a number of transformations at different energies. We now indicate the structure of the theory for the seven nontrivial cases considered earlier.

For case (1) the matrix $M$ has just a single element when a pair of transformations are used at a single energy and is given by

$$M_{11} = 1 + \lambda \int_0^r \left( \psi^{(0)}(x) \right)^2 dx .$$  
(280)

In terms of a second matrix $\tilde{M}$ given by

$$\tilde{M} = \left( \begin{array}{cc} M_{11} & \int_0^r \psi^{(0)}(x) \psi^{(m)}(x) dx \\ \psi^{(0)}(r) & \psi^{(m)}(r) \end{array} \right) ,$$  
(281)

the eigenfunction relation given in Eq. (239) may be expressed in the form

$$\tilde{\psi}_{1,2}^{(m)} = \frac{\det \tilde{M}}{\det M} .$$  
(282)

For case (2) the matrix $M$ has element

$$M_{11} = \int_0^r \left( \psi^{(0)}(x) \right)^2 dx ,$$  
(283)

and $\tilde{M}$ is given by

$$\tilde{M} = \left( \begin{array}{cc} M_{11} & \int_0^r \psi^{(0)}(x) \psi^{(m)}(x) dx \\ \psi^{(0)}(r) & \psi^{(m)}(r) \end{array} \right) .$$  
(284)

The eigenfunction relation in Eq. (244) can be written in terms of these matrices as

$$\tilde{\psi}_{1,3}^{(m)} = \frac{\det \tilde{M}}{\det M} .$$  
(285)

For case (3) the matrix $M$ has element

$$M_{11} = \int_r^\infty \left( \psi^{(0)}(x) \right)^2 dx ,$$  
(286)

and $\tilde{M}$ is given by

$$\tilde{M} = \left( \begin{array}{cc} M_{11} & \int_r^\infty \psi^{(0)}(x) \psi^{(m)}(x) dx \\ \psi^{(0)}(r) & \psi^{(m)}(r) \end{array} \right) .$$  
(287)
The eigenfunction relation in Eq. (247) may then be written in the form

\[ \tilde{\psi}_{1,4}^{(m)} = \frac{\det \tilde{M}}{\det M}. \]  

(288)

For case (4) \( M \) is given by

\[ M_{11} = 1 + \int_0^r (\phi(x, \tilde{E}))^2 \, dx, \]

(289)

and the matrix \( \tilde{M} \) has the form

\[ \tilde{M} = \left( \begin{array}{cc} M_{11} & \int_0^r \phi(x, \tilde{E}) \psi^{(m)}(x) \, dx \\ \phi(r, \tilde{E}) & \psi^{(m)}(r) \end{array} \right). \]

(290)

The eigenfunction relation in Eq. (259) may be given in the form

\[ \tilde{\psi}_{3,2}^{(m)} = \frac{\det \tilde{M}}{\det M}. \]

(291)

For case (5)

\[ M_{11} = 1 + \int_r^\infty (\eta(x, \tilde{E}))^2 \, dx, \]

(292)

and

\[ \tilde{M} = \left( \begin{array}{cc} M_{11} & \int_r^\infty \eta(x, \tilde{E}) \psi^{(m)}(x) \, dx \\ \eta(r, \tilde{E}) & \psi^{(m)}(r) \end{array} \right). \]

(293)

The eigenfunction relation in Eq. (270) may be written in the form

\[ \tilde{\psi}_{4,2}^{(m)} = \frac{\det \tilde{M}}{\det M}. \]

(294)

For case (6)

\[ M_{11} = \int_0^r (\phi(x, \tilde{E}))^2 \, dx, \]

(295)

and

\[ \tilde{M} = \left( \begin{array}{cc} M_{11} & \int_0^r \phi(x, \tilde{E}) \psi^{(m)}(x) \, dx \\ \phi(r, \tilde{E}) & \psi^{(m)}(r) \end{array} \right). \]

(296)

The eigenfunction relation in Eq. (273) may be written in the form

\[ \tilde{\psi}_{3,3}^{(m)} = \frac{\det \tilde{M}}{\det M}. \]

(297)

Similarly for case (8)

\[ M_{11} = \int_r^\infty (\eta(x, \tilde{E}))^2 \, dx, \]

(298)
and
\[
\tilde{M} = \left( \begin{array}{c}
M_{11} & \int_{r}^{\infty} \eta(x, \tilde{E}) \psi^{(m)}(x) dx \\
\eta(r, \tilde{E}) & \psi^{(m)}(r)
\end{array} \right).
\]

The eigenfunction relation in Eq. (277) may be given in the form
\[
\tilde{\psi}^{(m)}_{4,4} = \frac{\det \tilde{M}}{\det M}.
\]

5.3. Examples

The theory outlined in the last section may be illustrated with some exactly solvable examples. To illustrate the first three cases we consider the potential
\[
V(r) = -3 \text{sech}^2 r,
\]
which supports a single bound state at \( E = -1/2 \) with eigenfunction
\[
\psi^{(0)}(r) = \sqrt{3} Z \sqrt{1 - Z^2}, \quad Z = \tanh r.
\]
It is easy to show that
\[
\int_{r}^{\infty} \left( \psi^{(0)}(x) \right)^2 dx = 1 - \tanh^3 r, \quad (303)
\]
\[
\int_{0}^{r} \left( \psi^{(0)}(x) \right)^2 dx = \tanh^3 r. \quad (304)
\]

By using the procedure for case (1) the family of phase-equivalent potentials in Eq. (233) can be given as
\[
\tilde{V}_{1,2} = -3 \frac{\text{sech}^2 r}{(1 + \lambda \tanh^3 r)} \left( 1 + 2 \lambda \tanh r \text{sech}^2 r - \lambda^2 \tanh^4 r \right). \quad (305)
\]
For values of \( \lambda > -1 \), the family of potentials \( \tilde{V}_{1,2} \) have a single bound state at the same energy but differ in the normalization constant of the ground state eigenfunction. \( \lambda = 0 \) corresponds to the original potential \( V(r) \).

The method used for case (2) leads to the potential in Eq. (241) which in this example becomes
\[
\tilde{V}_{1,3}(r) = \frac{3}{\sinh^2 r}, \quad (306)
\]
which is repulsive singular potential which supports no bound states and is phase-equivalent to \( V(r) \).

The method used for case (3) leads to the potential in Eq. (246) which in this example becomes
\[
\tilde{V}_{1,4}(r) = -3 \left( \frac{\text{sech}^2 r}{1 + \tanh r + \tanh^2 r} \right)^2, \quad (307)
\]
which is an attractive non-singular potential which supports no bound states but has the phase shifts altered as specified by Eq. (248).

We next examine \( V = 0 \) and the transformation energy \( \tilde{E} = -\tilde{\gamma}^2/2 \). The regular solution is

\[
\phi = \sinh \tilde{\gamma}r, \quad \lim_{r \to 0} \phi = \tilde{\gamma}r, \quad \lim_{r \to \infty} \phi \sim \exp(+\tilde{\gamma}r),
\]

and the Jost solution is

\[
\eta = \exp(-\tilde{\gamma}r), \quad \lim_{r \to 0} \eta = 1, \quad \lim_{r \to \infty} \eta = \exp(-\tilde{\gamma}r).
\]

The potential arising from a \( T_3 \) transformation is

\[
\tilde{V}_3 = \frac{\tilde{\gamma}^2}{\sinh^2 \tilde{\gamma}r},
\]

which is a repulsive potential. The integral relation

\[
\int_0^r \sinh^2 \tilde{\gamma}rdx \equiv F = \frac{\sinh 2\tilde{\gamma}r - 2\tilde{\gamma}r}{4\tilde{\gamma}},
\]

may be used to find the potential in Eq. (253) in the form

\[
\tilde{\tilde{V}}_{3,2} = \frac{\sinh^4 \tilde{\gamma}r}{(\cot \alpha + F)^2} - \frac{\tilde{\gamma} \sinh 2\tilde{\gamma}r}{\cot \alpha + F},
\]

which is a non-singular family of phase-equivalent potentials for \( 0 < \alpha < \pi/2 \) with normalized ground state eigenfunctions given by

\[
\tilde{\psi}_{3,2} = \sqrt{\sin \alpha \cos \alpha} \frac{\sinh \tilde{\gamma}r}{\cos \alpha + F \sin \alpha}.
\]

The potential after the \( T_4 \) transformation is \( \tilde{V}_4 = 0 \). Using the integral relation

\[
\int_r^\infty (\eta(x))^2 \, dx = \frac{\exp(-2\tilde{\gamma}r)}{2\tilde{\gamma}},
\]

the potential in Eq. (264) in this example becomes

\[
\tilde{\tilde{V}}_{4,2} = -\tilde{\gamma}^2 \text{sech}^2 (\tilde{\gamma}r + \varepsilon), \quad 2\varepsilon = \ln (2\tilde{\gamma} \cot \beta),
\]

which is also a family of phase-equivalent potentials for \( 0 < \beta < \pi/2 \), with ground state eigenfunctions given by

\[
\tilde{\psi}_{4,2} \sim \frac{\exp(-\tilde{\gamma}r)}{\cos \beta + \sin \beta \exp(-2\tilde{\gamma}r)}.
\]

The potential after two \( T_3 \) transformations given by Eq. (272) in this example is

\[
\tilde{\tilde{V}}_{3,3} = \frac{\sinh^4 \tilde{\gamma}r}{F^2} - \frac{\tilde{\gamma}}{F} \sinh 2\tilde{\gamma}r,
\]

where \( F \) is given by Eq. (311).

The potential after two \( T_4 \) transformation given by Eq. (276) in this example gives \( \tilde{\tilde{V}}_{4,4} = 0 \). This completes the list of transformed potentials after two \( T \) transformations.
5.4. Summary

We have studied the properties of potentials and eigenfunctions arising from two successive supersymmetric transformations performed at the same energy. We have illustrated the procedure using an exactly solvable example. The method developed in this lecture may be extended to eliminate or add an arbitrary number of states or alter the phase shifts without altering the spectrum. The generalized procedure for accomplishing the above mentioned tasks will be the subject of the next lecture.

6. SEQUENCE OF SUSY TRANSFORMATIONS, REGULAR AND SINGULAR POTENTIALS AND APPLICATIONS

The procedure discussed in the previous lectures for the modification of the spectrum of a potential corresponding to a radial Schrödinger equation by a two-step procedure may be generalized to discuss the removal or addition of an arbitrary number of bound states to a given spectrum or to generate the phase equivalent families corresponding to given potentials. The singular potentials arising from the two-step procedure have proved useful in understanding the relation between deep and shallow potentials both of which have been used with some success in describing a variety of scattering problems in Nuclear Physics. Some examples will be discussed.

6.1. Elimination of m bound states and non-singular potentials

In §5 it was shown that a transformation of the type $T_1$ followed by a transformation of the type $T_4$ may be used to remove the ground state of a potential $V(r)$ for a definite partial wave without altering the spectral density and the normalization constants of all the excited state eigenfunctions of $V(r)$. The phase shifts for positive energies will, however, be altered in a definite manner. The resulting potential and the new eigenfunctions were given in terms of the matrices $M$ and $\tilde{M}$ defined by

$$ M_{11} = \int_r^\infty \left( \psi^{(0)}(x) \right)^2 dx, $$

and

$$ \tilde{M}^{(n)} = \left( \int_r^\infty \left( \psi^{(0)}(x) \right)^2 dx \int_r^\infty \psi^{(0)}(x) \psi^{(n)}(x) dx \right), \quad n = 1, 2, \ldots, $$

in the form

$$ \tilde{V}_{1,4}(r) = V(r) - \frac{d^2}{dr^2} \ln(\det M) $$

and

$$ \tilde{\psi}_{1,4}^{(n)}(r) = \frac{\det \tilde{M}^{(n)}}{\det M}, \quad n = 1, 2, \ldots. $$
The phase shifts for positive energies were related by
\[ \tilde{\delta}_{1,4}(l,k) = \delta(l,k) - 2\tan^{-1}\left(\frac{\gamma(0)}{k}\right). \] (322)

It was also shown that the two transformations \( T_4T_1 \) taken together do not introduce any additional singularities in the new potential.

The procedure used for eliminating a single bound state may be generalized to the case of elimination of an arbitrary number \( m \) of bound states as follows. Let \( V_0 \) be a potential with bound states given by
\[ V_0 : \begin{bmatrix} E^{(1)} & E^{(2)} & \ldots & E^{(n)} \\ \psi_0^{(1)} & \psi_0^{(2)} & \ldots & \psi_0^{(n)} \end{bmatrix}, \] (323)
determined from the solutions to the radial Schrödinger equation
\[ \left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} + V_0(r) - E^{(j)}\right] \psi_0^{(j)}(r) = 0, \quad j = 1, 2, \ldots, n. \] (324)
The eigenfunction set in Eq. (323) may be taken to be an orthonormal set. The incomplete overlap integrals may be defined by
\[ F_{jk} = \int_{r_0}^{\infty} \psi_0^{(j)}(x)\psi_0^{(k)}(x)dx, \quad j, k = 1, 2, \ldots, n, \] (325)
\[ G_{jk} = \int_{0}^{r} \psi_0^{(j)}(x)\psi_0^{(k)}(x)dx, \quad j, k = 1, 2, \ldots, n. \] (326)
The two matrices \( F \) and \( G \) are related by \( F + G = I \) where \( I \) is the unit matrix of dimension \([n,n] \). By using the orthonormal property of the eigenfunctions and the Wronskian relation between two solutions of the Schrödinger equation in the same potential at two different energies, \( F_{jk} \) and \( G_{jk} \) can also be written as
\[ F_{jk} = -G_{jk} = \frac{1}{2(E^{(k)} - E^{(j)})} \left( \psi_0^{(j)}(r) \frac{d}{dr} \psi_0^{(k)}(r) - \psi_0^{(k)}(r) \frac{d}{dr} \psi_0^{(j)}(r) \right), \quad j \neq k. \] (327)

Let \( V_m \) be the potential generated from \( V_0 \) by the application of a sequence consisting of \( m \) pairs of \( T_1 \) and \( T_4 \) transformations for eliminating the lowest \( m \) bound states of \( V_0 \) so that the bound states of \( V_m \) are given by
\[ V_m : \begin{bmatrix} E^{(m+1)} & E^{(m+2)} & \ldots & E^{(n)} \\ \psi_{m}^{(m+1)} & \psi_{m}^{(m+2)} & \ldots & \psi_{m}^{(n)} \end{bmatrix}. \] (328)
Then it may be shown by a generalization of the method for eliminating a single bound state that in terms of a matrix \( M \) with elements given by
\[ M_{jk} = F_{jk}, \quad j, k = 1, 2, \ldots, m, \] (329)
the potential $V_m$ with $n - m$ bound states may be given as

$$V_m = V_0 - \frac{d^2}{dr^2} \ln (\det M) \ . \ \ \ \ \ \ \ (330)$$

The eigenfunctions of $V_m$ are related to those of $V_0$ by

$$\psi_m^{(i)} = \frac{\det \tilde{M}^{(i)}}{\det M}, \quad i = m + 1, m + 2, \ldots, n, \ \ \ \ \ \ \ (331)$$

where the elements of $\tilde{M}^{(i)}$ are

$$\tilde{M}^{(i)}_{jk} = M_{jk}, \quad j, k = 1, 2, \ldots, m, \ \ \ \ \ \ \ (332)$$

Eqs. (329)-(332) provide a recipe for constructing a new potential by removing $m$ bound states and which does not have any additional singularities present which are not already present in $V_0$. This recipe is the same as the recipe based on the Gelfand-Levitan equations. The phase shifts in the new potential $V_m$ for positive energies are related to the phase shifts in $V_0$ by

$$\tilde{\delta}_m(l, k) = \delta(l, k) - 2 \sum_{j=1}^{m} \tan^{-1}\left(\frac{\gamma^{(j)}}{k}\right), \ \ \ \ \ \ \ (333)$$

where $\gamma^{(j)} = \sqrt{-2E^{(j)}}$. In the limit $k \to 0$ the phase shift decreases by $m\pi$ in agreement with the general result for a potential with $m$ fewer bound states. In the limit $k \to \infty$ the phase shifts in the two potentials are identical.

The expression for the new eigenfunctions in Eq. (331) may also be given in other forms by expanding the determinant in the numerator and using the definition of the elements of the inverse of a matrix as the ratio of two determinants. This leads to the expression

$$\psi_m^{(i)} = \psi_0^{(i)} - \sum_{k=1}^{m} \sum_{j=1}^{m} \psi_0^{(k)}(M^{-1})_{kj}F_{ji}, \quad i = m + 1, m + 2, \ldots, n . \ \ \ \ \ \ \ (334)$$

Since the index $j$ in the above equation can only take values in the range $1 \leq j \leq m$ and the index $i$ takes values $m < i \leq n$ the matrix element $F_{ji}$ may be replaced by $-G_{ji}$ so that Eq. (334) may also be given in the form

$$\psi_m^{(i)} = \psi_0^{(i)} + \sum_{k=1}^{m} \sum_{j=1}^{m} \psi_0^{(k)}(M^{-1})_{kj}G_{ji}, \quad i = m + 1, m + 2, \ldots, n . \ \ \ \ \ \ \ (335)$$
It may be shown that in this form the above equation may be extended to allow the index $i$ to take all values in the range $[1,n]$ to define a set of solutions in $V_m$ at the energies at which the bound states have been removed. It can be shown that these non-normalizable solutions in $V_m$ at the energies corresponding to the removed bound states satisfy

$$\psi_m^{(j)} = \sum_{k=1}^{m} \psi_0^{(k)} (M^{-1})_{kj}.$$  \hspace{1cm} (336)

In terms of the solutions to the linear equations

$$\sum_{j=1}^{m} M_{kj} \psi_m^{(j)} = \psi_0^{(j)}, \quad k = 1, 2, \ldots, m.$$  \hspace{1cm} (337)

the normalizable bound state eigenfunctions of $V_m$ may be given as

$$\psi_m^{(i)} = \psi_0^{(i)} - \sum_{k=1}^{m} \psi_m^{(k)} F_{ki}, \quad i = m+1, m+2, \ldots, n.$$  \hspace{1cm} (338)

Thus by constructing the matrix $M$ and solving the linear equations (337) to find $\psi_0^{(k)}$ once it is possible to find all the eigenfunctions of $V_m$ using Eq. (338). It may be shown that the expression for the new potential in Eq. (330) can be further simplified by expanding the logarithmic derivative to get

$$\frac{d}{dr} \ln (\det M) = \sum_{k=1}^{m} \psi_0^{(k)}(r) \psi_m^{(k)}(r),$$  \hspace{1cm} (339)

so that the modified potential may be expressed in the form

$$V_m(r) = V_0(r) - \frac{d}{dr} \left( \sum_{k=1}^{m} \psi_0^{(k)}(r) \psi_m^{(k)}(r) \right).$$  \hspace{1cm} (340)

In this form the potential arising from the removal of $m$ bound states is expressed as a symmetric function of the solutions in the old and new potentials at the energies at which the bound states have been removed.

### 6.2. Elimination of $m$ bound states and singular potentials

In §5 it was shown that a transformation of the type $T_1$ followed by a transformation of the type $T_3$ may be used to remove the ground state of a potential $V(r)$ without altering the phase shifts for positive energies. However, the normalization constants of the eigenstates and the spectral density are altered at the end of the two transformations. The resulting potential and the new eigenfunctions were given in terms of the matrices $\mathcal{N}$ and $\tilde{\mathcal{N}}$ defined by

$$N_{11} = \int_0^r \left( \psi^{(0)}(x) \right)^2 dx,$$  \hspace{1cm} (341)
\[ \tilde{N}^{(n)} = \left( \int_0^r \left( \psi_0^{(0)}(x) \right)^2 dx, \int_0^r \psi_0^{(0)}(x) \psi^{(n)}(x) dx \right), \quad n = 1, 2, \ldots, \tag{342} \]

in the form

\[ \tilde{V}_{1,3}(r) = V(r) - \frac{d^2}{dr^2} \ln(\det N), \tag{343} \]
\[ \tilde{\psi}_{1,3}^{(n)}(r) = \frac{\det \tilde{N}^{(n)}}{\det N}, \quad n = 1, 2, \ldots. \tag{344} \]

It was also shown that \( \tilde{V}_{1,3} \) is a singular potential with a short range repulsive \( r^{-2} \) character which as \( r \to 0 \) is equivalent to \( I \to (I + 2) \) in the centrifugal part of the potential. It was also shown that the phase shift decreases by \( \pi \) for all energies and therefore the phase shifts in the two potentials are the same within modulo \( \pi \).

The procedure used for eliminating a single bound state may be generalized to the case of elimination of an arbitrary number \( m \) of bound states. Let \( \bar{V}_m \) be the potential generated from \( V_0 \) by the application of a sequence consisting of \( m \) pairs of \( T_1 \) and \( T_3 \) transformations for eliminating the lowest \( m \) bound states of \( V_0 \) so that the bound states of \( \bar{V}_m \) are given by

\[ \bar{V}_m : \begin{bmatrix} E^{(m+1)} & E^{(m+2)} & \ldots & E^{(n)} \\ \psi^{(m+1)}_m & \psi^{(m+2)}_m & \ldots & \psi^{(n)}_m \end{bmatrix}. \tag{345} \]

It may be shown by a generalization of the method for eliminating a single bound state that in terms of a matrix \( N \) with elements given by

\[ N_{jk} = \int_0^r \psi_0^{(j)}(x) \psi_0^{(k)}(x) dx = G_{jk}, \quad j, k = 1, 2, \ldots, m, \tag{346} \]

the potential \( \bar{V}_m \) with \( n - m \) bound states may be given as

\[ \bar{V}_m = V_0 - \frac{d^2}{dr^2} \ln(\det N). \tag{347} \]

It is clear that the elements of the matrix \( N \) and the matrix \( M \) considered in the previous section are related by \( M + N = I \) where \( I \) is the unit matrix of dimension \( (m, m) \). This means that the elements of the two matrices are related by

\[ N_{jk} = -M_{jk}, \quad j \neq k, \quad N_{jj} = 1 - M_{jj}. \tag{348} \]

The eigenfunctions of \( \bar{V}_m \) are related to the eigenfunctions of \( V_0 \) by

\[ \psi_m^{(j)} = \frac{\det \tilde{N}^{(i)}}{\det N}, \quad i = m + 1, m + 2, \ldots, n, \tag{349} \]
where the elements of $\tilde{N}^{(i)}$ are

\[
\begin{align*}
\tilde{N}_{jk}^{(i)} &= N_{jk}, & j, k &= 1, 2, \ldots, m, \\
\tilde{N}_{j,m+1}^{(i)} &= \int_0^r \psi_0^{(j)}(x)\psi_0^{(i)}(x)dx = G_{ji}, & j &= 1, 2, \ldots, m, \\
\tilde{N}_{m+1,k}^{(i)} &= \psi_0^{(k)}(r), & k &= 1, 2, \ldots, m, \\
\tilde{N}_{m+1,m+1}^{(i)} &= \psi_0^{(i)}(r), & i &= m+1, m+2, \ldots, n.
\end{align*}
\] (350)

Eqs. (346)-(350) provide a recipe for constructing a new potential by removing $m$ bound states without altering the phase shifts but altering the spectral density. In fact the phase shifts decrease by $m\pi$ for all positive energies but the two phase shifts are the same within modulo $\pi$. The resulting potential is singular and has a repulsive character as $r \to 0$. It can be shown that

\[
\lim_{r \to 0} \bar{V}_m = \frac{(l+2m)(l+2m+1)}{2r^2}, \quad \lim_{r \to \infty} \bar{V}_m = \frac{l(l+1)}{2r^2},
\] (351)

which shows that the repulsive singularity at the origin rises rapidly as the number of bound states removed increases.

The expression for the new eigenfunctions in Eq. (349) may also be given in other forms by expanding the determinant in the numerator and using the definition of the elements of the inverse of a matrix as the ratio of two determinants. This leads to the expression

\[
\bar{\psi}_m^{(i)} = \psi_0^{(i)} + \sum_{k=1}^{m} \sum_{j=1}^{m} \psi_0^{(k)} (N^{-1})_{kj} F_{ji}, \quad i = m+1, m+2, \ldots, n.
\] (352)

It may be shown that the non-normalizable solutions in $\bar{V}_m$ at the energies corresponding to the removed bound states may also be defined by Eq. (352) by allowing the index $i$ to take values in the range $(1, m)$. Then by using $F_{jk} = M_{jk}$ and the matrix relation in Eq. (344) the above equation may be further simplified to give

\[
\psi_m^{(j)} = \sum_{k=1}^{m} \psi_0^{(k)} (N^{-1})_{kj}, \quad j = 1, 2, \ldots, m.
\] (353)

Thus in terms of the solutions to the linear equations

\[
\sum_{k=1}^{m} N_{jk} \psi_m^{(k)} = \psi_0^{(j)}, \quad j = 1, 2, \ldots, m,
\] (354)

the bound state eigenfunctions in $\bar{V}_m$ are

\[
\psi_m^{(i)} = \psi_0^{(i)} - \sum_{k=1}^{m} \psi_m^{(k)} G_{ki}, \quad i = m+1, m+2, \ldots, n.
\] (355)
Thus by constructing the matrix $N$ and solving the linear equations (354) to find $\psi(m)$ once it is possible to find all the eigen functions of $V_m$ using Eq. (355). It may be shown that the expression for the new potential in Eq. (347) can be further simplified by expanding the logarithmic derivative to get

$$\frac{d}{dr} \ln(\det N) = \sum_{k=1}^{m} \psi_0^{(k)}(r) \bar{\psi}_m^{(k)}(r),$$

so that the modified potential may be expressed in the form

$$V_m(r) = V_0(r) - \frac{d}{dr} \left( \sum_{k=1}^{m} \psi_0^{(k)}(r) \bar{\psi}_m^{(k)}(r) \right).$$

In this form the potential arising from the removal of $m$ bound states is expressed as a symmetric function of the solutions in the old and new potentials at the energies at which the bound states have been removed.

So far we have discussed in detail the procedure for removing $m$ bound states to produce singular or non-singular potentials. It is possible to give similar generalizations of the other cases discussed in lecture IV to provide recipes for adding $m$ bound states to produce singular and non-singular potentials or find extended families of potentials with the same spectrum but differing in the normalization constants of the eigenfunctions and/or differing in the phase shifts. We have concentrated on a detailed discussion of the case of the removal bound states because this has found some applications in Nuclear Physics. We discuss the application of these ideas to certain problems in Nuclear Physics in the next section.

### 6.3. Deep and shallow potentials in nuclear physics

In Nuclear Physics both deep and shallow potentials have been used to describe nucleus-nucleus interactions both of which fit the set of known experimental data such as energy levels equally well. However, their predictions can differ for other energy domains where the experiments have not been performed yet or where the properties studied involve the explicit use of eigenfunctions to calculate matrix element such as the calculation of electromagnetic transition probabilities or radiative-capture cross sections. The question of the relation between deep and shallow potentials and their appropriateness for the study of nuclear interactions has been a controversial issue for a long time.

Microscopic models making use of fully antisymmetric scattering eigenfunctions which provide phase shifts that satisfy a modified Levinson’s theorem [49, 50, 51] have been developed [52] which lead to a description of internuclear interactions in terms of deep local potentials. These deep potentials have a number of unphysical bound states which simulate the forbidden states of a microscopic approach to the study of nucleus-nucleus interaction and are needed to provide phase shifts which agree with the generalized Levinson theorem in the high energy limit. Deep potentials which accurately fit the data for $\alpha + \alpha$ (Buck et al. [52]), $\alpha + ^{16}\text{O}$ (Michel et al. [53]) and $\alpha + ^{40}\text{Ca}$ (Michel...
and Vanderpoorten [54]) have been constructed. The real part of these potentials either do not depend or weakly depend on energy and angular momentum.

In contrast shallow potentials which have no unphysical bound states in their spectra but which have strong angular momentum dependence and fit the same experimental data equally well have also been constructed for some systems such as $\alpha + \alpha$ (Ali and Bodmer [55]). The bound states of the shallow potentials can be interpreted as good approximations to the actual physical states of the fused nucleus. Michel and Reidmeister [56] showed that it is possible to construct shallow potentials which are singular when $r \to 0$ but are able to produce phase shifts at high energies which are in agreement with the microscopic phase shifts and derived phenomenological singular potentials which are phase equivalent to the real part of the $\alpha + ^{16}O$ deep potential.

The relation between inverse scattering theory and the algebra of supersymmetry studied in these lectures sheds light on the relation between deep and shallow potentials both of which explain the same data well. The method discussed in §6.3 to remove a number of bound states from a deep potential $V_0$ and generate a singular potential $\bar{V}_m$ with $m$ fewer bound states but which is phase equivalent to the deep potential has been used for a number of nuclear systems by Baye [57]. We now consider the $\alpha + \alpha$ scattering discussed by Baye. The starting potential $V_0$ for this system is the two parameter potential of Buck, Friedrich and Wheatley [52]. The s-wave potential $V_0$ has three bound states one of which is at .092 MeV and corresponds to a physical state, (viz) the ground state of $Be^8$, and two other bound states at -72.8 MeV and -25.9 MeV which are unphysical. The two-step SUSY procedure for removing bound states discussed in §6.3 is used to first remove the bound state at -72.8 MeV to produce the singular potential $\bar{V}_1$ which has two bound states. The potential arising from the removal of both the unphysical bound states of $V_0$ is the potential $\bar{V}_2$ which is singular but has only one bound state which is the physical bound state. $V_0$, $\bar{V}_1$ and $\bar{V}_2$ have identical phase shifts for all energies and are phase equivalent. Fig. 8 shows the results of such a calculation similar to the one performed by Baye [57]. The $\bar{V}_2$ so constructed is the shallow but singular potential which is the phase equivalent partner for s-waves to the deep potential used by Buck et al.. The deep potential has one unphysical bound state for $l = 2$ and no unphysical bound states for $l = 4$. The SUSY procedure was used by Baye to remove the unphysical bound states and the resulting potentials for $l = 0, 2$ and 4 partial waves denoted by $\bar{V}_2^{(0)}$, $\bar{V}_2^{(2)}$ and $\bar{V}_2^{(4)}$, respectively, were compared with the shallow $\alpha + \alpha$ potentials used by Ali and Bodmer [55] and good agreement was found.

Thus the calculation by Baye [57] has demonstrated that the singular, shallow, $l$-dependent Ali-Bodmer potentials for $\alpha + \alpha$ scattering is an approximate supersymmetric partner to the deep, $l$-independent potential of Buck, Friedrich and Wheatley.

Sparenberg and Baye [58] have analysed inverse scattering with singular potentials in detail using the supersymmetric approach. They have shown that by using potentials with a $r^{-2}$ singularity at the origin, the inverse scattering problem at fixed orbital angular momentum $l$ can be decomposed into two parts. In the first step the phase shift information is used to construct a singular potential without a bound state. When the phase shift at higher energies is smaller than the phase shift at zero energy, the effective potential has a repulsive core of the form $\nu(\nu + 1)/(2\nu^2)$ where $\nu$ is larger than $l$. In the second step the bound states are added without modifying the phase shift.
Singular $\alpha + \alpha$ potentials for $l = 0$ generated from the deep potential $V_0$ of Buck, Friedrich and Wheatley [52] by the successive elimination of the two unphysical bound states of $V_0$. $\bar{V}_2$ is phase equivalent to $V_0$ and $\bar{V}_1$. The removed bound states are represented by horizontal bars.

Sparenberg and Baye have used this approach to invert the experimental phase shifts of neutron-proton elastic scattering in the $^1S_0$ state to obtain singular potential with $\nu = 1$. Andrianov, Borisov and Ioffe [59] have investigated the possibility of a supersymmetric description of two half-integer spin particles by studying the supersymmetric aspect of $NN$ and $N\bar{N}$ scattering in the one-pion exchange approximation. Andrianov, Cannata, Ioffe and Nishanidze [60] have studied matrix Hamiltonians using a SUSY approach to hidden symmetries and have analysed coupled channel problems using this approach. Sparenberg and Baye [61] have used SUSYQM to find coupled-channel potentials which are phase equivalent and have removed the non-physical state of the deep $^3S_1$-$^3D_1$ neutron-proton Moscow potential and transformed it into a shallow potential with a repulsive core. There are many other examples of such applications of singular potentials in Nuclear Physics.

7. CONCLUSIONS

In this course of lectures we have shown that starting from the simple idea of factorising second order differential equations of a single variable it is possible to find a related differential equation whose solutions are closely related to those of the original equation.
The relationship between the spectra and eigenfunctions of the two differential operators are similar to those between the components of a supersymmetric doublet. The flexibility available in the choice of factorization enables the manipulation of the spectrum of the operators in desired fashion. We have shown that using the factorization method it is possible to add or remove bound states or maintain the same spectrum but alter the normalization constants of the eigenfunctions.

We have used these ideas to study the radial Schrödinger equation in detail. If the potential of interaction is known then it is possible to find the eigenvalue spectrum, eigenfunctions and the phase shifts for positive energies by solving the radial Schrödinger equation for a given partial wave. But if the potential of interaction is not known accurately from first principles then one must use the spectral data like phase shifts and energy levels to work backwards to find the potential. This is the subject of study of inverse scattering theory. To determine the potential uniquely all the energy levels, the normalization constants associated with the eigenfunctions of all the energy levels and the phase shifts at all positive energies from 0 to $\infty$ must be known. Such a complete list of data seldom exists and often phenomenological potentials with widely differing properties in terms of their behaviour as a function of $r$ are used. In these lectures we have shown that the concept of supersymmetric pairing enables the construction of inverse scattering theory from some basic building blocks. We have shown that it is possible to construct families of potentials with related phase shifts and other spectral properties. Such constructions have enabled the establishment of links between potentials with startlingly dissimilar structure which nevertheless have closely related spectral properties. Even though the same properties could be found from the usual methods of inverse scattering theory based on Gelfand-Levitan and Marchenko equations the method based on the methods of SUSYQM has a particular simplicity and economy of expression and opens up the possibility of similar constructions for other inverse problems arising in other contexts. The method based on the supersymmetric construction has now been extended to the study of coupled differential equations which has proved useful in the study of a variety of coupled-channel problems in Nuclear Physics.

We have shown that the concept of supersymmetric pairing may be used to elucidate the spectrum of the Dirac equation for an attractive Coulomb potential. We have also shown that the Backlund transformation which provides a link between the $N$ and $N+1$ soliton solutions of the KdV hierarchy of non-linear differential equations is closely related to a supersymmetric transformation which links the potential $V_N$ with $N$ bound states to another potential $V_{N+1}$ which has an additional bound state below the ground state of $V_N$.

The construction of singular potentials using suitable pairs of supersymmetric transformations and their success in elucidating the relationship between different model potentials in Nuclear Physics is an outstanding example of the usefulness of SUSYQM in a real physical context.

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